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COMMON CORE STATE STANDARDS INITIATIVE
PREPARING AMERICA'S STUDENTS FOR COLLEGE & CAREER

Colorado Academic STANDARDS

High School

Mathematics

Colorado Academic Standards in Mathematics and The Common Core State Standards for Mathematics

On December 10, 2009, the Colorado State Board of Education adopted the revised Mathematics Academic Standards, along with academic standards in nine other content areas, creating Colorado's first fully aligned preschool through high school academic expectations. Developed by a broad spectrum of Coloradans representing Pre-K and K-12 education, higher education, and business, utilizing the best national and international exemplars, the intention of these standards is to prepare Colorado schoolchildren for achievement at each grade level, and ultimately, for successful performance in postsecondary institutions and/or the workforce.

Concurrent to the revision of the Colorado standards was the Common Core State Standards (CCSS) initiative, whose process and purpose significantly overlapped with that of the Colorado Academic Standards. Led by the Council of Chief State School Officers (CCSSO) and the National Governors Association (NGA), these standards present a national perspective on academic expectations for students, Kindergarten through High School in the United States.

Upon the release of the Common Core State Standards for Mathematics on June 2, 2010, the Colorado Department of Education began a gap analysis process to determine the degree to which the expectations of the Colorado Academic Standards aligned with the Common Core. The independent analysis proved a nearly 95% alignment between the two sets of standards. On August 2, 2010, the Colorado State Board of Education adopted the Common Core State Standards, and requested the integration of the Common Core State Standards and the Colorado Academic Standards.

In partnership with the dedicated members of the Colorado Standards Revision Subcommittee in Mathematics, this document represents the integration of the combined academic content of both sets of standards, maintaining the unique aspects of the Colorado Academic Standards, which include personal financial literacy, 21st century skills, school readiness competencies, postsecondary and workforce readiness competencies, and preschool expectations. The result is a world-class set of standards that are greater than the sum of their parts.

The Colorado Department of Education encourages you to review the Common Core State Standards and the extensive appendices at www.corestandards.org. While all the expectations of the Common Core State Standards are embedded and **coded with CCSS:** in this document, additional information on the development and the intentions behind the Common Core State Standards can be found on the website.

Colorado Academic Standards
Mathematics Standards

"Pure mathematics is, in its way, the poetry of logical ideas."
Albert Einstein

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"If America is to maintain our high standard of living, we must continue to innovate. We are competing with nations many times our size. We don't have a single brain to waste. Math and science are the engines of innovation. With these engines we can lead the world. We must demystify math and science so that all students feel the joy that follows understanding."
Dr. Michael Brown, Nobel Prize Laureate

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In the 21st century, a vibrant democracy depends on the full, informed participation of all people. We have a vast and rapidly growing trove of information available at any moment. However, being *informed* means, in part, using one's sense of number, shape, data and symbols to organize, interpret, make and assess the validity of claims about quantitative information. In short, informed members of society know and do mathematics.

Mathematics is indispensable for understanding our world. In addition to providing the tools of arithmetic, algebra, geometry and statistics, it offers a way of thinking about patterns and relationships of quantity and space and the connections among them. Mathematical reasoning allows us to devise and evaluate methods for solving problems, make and test conjectures about properties and relationships, and model the world around us.

Standards Organization and Construction

As the subcommittee began the revision process to improve the existing standards, it became evident that the way the standards information was organized, defined, and constructed needed to change from the existing documents. The new design is intended to provide more clarity and direction for teachers, and to show how 21st century skills and the elements of school readiness and postsecondary and workforce readiness indicators give depth and context to essential learning.

The "Continuum of State Standards Definitions" section that follows shows the hierarchical order of the standards components. The "Standards Template" section demonstrates how this continuum is put into practice.

The elements of the revised standards are:

Prepared Graduate Competencies: The preschool through twelfth-grade concepts and skills that all students who complete the Colorado education system must master to ensure their success in a postsecondary and workforce setting.

Standard: The topical organization of an academic content area.

High School Expectations: The articulation of the concepts and skills of a standard that indicates a student is making progress toward being a prepared graduate. *What do students need to know in high school?*

Grade Level Expectations: The articulation (at each grade level), concepts, and skills of a standard that indicate a student is making progress toward being ready for high school. *What do students need to know from preschool through eighth grade?*

Evidence Outcomes: The indication that a student is meeting an expectation at the mastery level. *How do we know that a student can do it?*

21st Century Skills and Readiness Competencies: Includes the following:

- ***Inquiry Questions:***
  Sample questions are intended to promote deeper thinking, reflection and refined understandings precisely related to the grade level expectation.

- ***Relevance and Application:***
  Examples of how the grade level expectation is applied at home, on the job or in a real-world, relevant context.

- ***Nature of the Discipline:***
  The characteristics and viewpoint one keeps as a result of mastering the grade level expectation.

Continuum of State Standards Definitions

Prepared Graduate Competency

Prepared Graduate Competencies are the P-12 concepts and skills that all students leaving the Colorado education system must have to ensure success in a postsecondary and workforce setting.

Standards

Standards are the topical organization of an academic content area.

P-8

Grade Level Expectations

Expectations articulate, at each grade level, the knowledge and skills of a standard that indicates a student is making progress toward high school.

What do students need to know?

Evidence Outcomes

Evidence outcomes are the indication that a student is meeting an expectation at the mastery level.

How do we know that a student can do it?

21st Century and PWR Skills

Inquiry Questions:
Sample questions intended to promote deeper thinking, reflection and refined understandings precisely related to the grade level expectation.

Relevance and Application:
Examples of how the grade level expectation is applied at home, on the job or in a real-world, relevant context.

Nature of the Discipline:
The characteristics and viewpoint one keeps as a result of mastering the grade level expectation.

High School

High School Expectations

Expectations articulate the knowledge and skills of a standard that indicates a student is making progress toward being a prepared graduate.

What do students need to know?

Evidence Outcomes

Evidence outcomes are the indication that a student is meeting an expectation at the mastery level.

How do we know that a student can do it?

21st Century and PWR Skills

Inquiry Questions:
Sample questions intended to promote deeper thinking, reflection and refined understandings precisely related to the grade level expectation.

Relevance and Application:
Examples of how the grade level expectation is applied at home, on the job or in a real-world, relevant context.

Nature of the Discipline:
The characteristics and viewpoint one keeps as a result of mastering the grade level expectation.

STANDARDS TEMPLATE

Content Area: NAME OF CONTENT AREA
Standard: The topical organization of an academic content area.

Prepared Graduates:

- The P-12 concepts and skills that all students who complete the Colorado education system must master to ensure their success in a postsecondary and workforce setting

High School and Grade Level Expectations

Concepts and skills students master:

Grade Level Expectation: High Schools: The articulation of the concepts and skills of a standard that indicates a student is making progress toward being a prepared graduate.

Grade Level Expectations: The articulation, at each grade level, the concepts and skills of a standard that indicates a student is making progress toward being ready for high school.

What do students need to know?

Evidence Outcomes	21st Century Skills and Readiness Competencies
Students can: Evidence outcomes are the indication that a student is meeting an expectation at the mastery level. *How do we know that a student can do it?*	**Inquiry Questions:** Sample questions intended to promote deeper thinking, reflection and refined understandings precisely related to the grade level expectation. **Relevance and Application:** Examples of how the grade level expectation is applied at home, on the job or in a real-world, relevant context. **Nature of the Discipline:** The characteristics and viewpoint one keeps as a result of mastering the grade level expectation.

Prepared Graduate Competencies in Mathematics

The prepared graduate competencies are the preschool through twelfth-grade concepts and skills that all students who complete the Colorado education system must master to ensure their success in a postsecondary and workforce setting.

Prepared graduates in mathematics:

- Understand the structure and properties of our number system. At their most basic level numbers are abstract symbols that represent real-world quantities
- Understand quantity through estimation, precision, order of magnitude, and comparison. The reasonableness of answers relies on the ability to judge appropriateness, compare, estimate, and analyze error
- Are fluent with basic numerical and symbolic facts and algorithms, and are able to select and use appropriate (mental math, paper and pencil, and technology) methods based on an understanding of their efficiency, precision, and transparency
- Make both relative (multiplicative) and absolute (arithmetic) comparisons between quantities. Multiplicative thinking underlies proportional reasoning
- Recognize and make sense of the many ways that variability, chance, and randomness appear in a variety of contexts
- Solve problems and make decisions that depend on understanding, explaining, and quantifying the variability in data
- Understand that equivalence is a foundation of mathematics represented in numbers, shapes, measures, expressions, and equations
- Make sound predictions and generalizations based on patterns and relationships that arise from numbers, shapes, symbols, and data
- Apply transformation to numbers, shapes, functional representations, and data
- Make claims about relationships among numbers, shapes, symbols, and data and defend those claims by relying on the properties that are the structure of mathematics
- Communicate effective logical arguments using mathematical justification and proof. Mathematical argumentation involves making and testing conjectures, drawing valid conclusions, and justifying thinking
- Use critical thinking to recognize problematic aspects of situations, create mathematical models, and present and defend solutions

Colorado Academic Standards
Mathematics

The Colorado academic standards in mathematics are the topical organization of the concepts and skills every Colorado student should know and be able to do throughout their preschool through twelfth-grade experience.

1. **Number Sense, Properties, and Operations**
   Number sense provides students with a firm foundation in mathematics. Students build a deep understanding of quantity, ways of representing numbers, relationships among numbers, and number systems. Students learn that numbers are governed by properties and understanding these properties leads to fluency with operations.

2. **Patterns, Functions, and Algebraic Structures**
   Pattern sense gives students a lens with which to understand trends and commonalities. Students recognize and represent mathematical relationships and analyze change. Students learn that the structures of algebra allow complex ideas to be expressed succinctly.

3. **Data Analysis, Statistics, and Probability**
   Data and probability sense provides students with tools to understand information and uncertainty. Students ask questions and gather and use data to answer them. Students use a variety of data analysis and statistics strategies to analyze, develop and evaluate inferences based on data. Probability provides the foundation for collecting, describing, and interpreting data.

4. **Shape, Dimension, and Geometric Relationships**
   Geometric sense allows students to comprehend space and shape. Students analyze the characteristics and relationships of shapes and structures, engage in logical reasoning, and use tools and techniques to determine measurement. Students learn that geometry and measurement are useful in representing and solving problems in the real world as well as in mathematics.

Modeling Across the Standards
Modeling links classroom mathematics and statistics to everyday life, work, and decision-making. Modeling is the process of choosing and using appropriate mathematics and statistics to analyze empirical situations, to understand them better, and to improve decisions. When making mathematical models, technology is valuable for varying assumptions, exploring consequences, and comparing predictions with data. Modeling is best interpreted not as a collection of isolated topics but rather in relation to other standards, specific modeling standards appear throughout the high school standards indicated by a star symbol (*).

Standards for Mathematical Practice
from
The Common Core State Standards for Mathematics

The Standards for Mathematical Practice have been included in the Nature of Mathematics section in each Grade Level Expectation of the Colorado Academic Standards. The following definitions and explanation of the Standards for Mathematical Practice from the Common Core State Standards can be found on pages 6, 7, and 8 in the Common Core State Standards for Mathematics. Each Mathematical Practices statement has been notated with (MP) at the end of the statement.

Mathematics | Standards for Mathematical Practice

The Standards for Mathematical Practice describe varieties of expertise that mathematics educators at all levels should seek to develop in their students. These practices rest on important "processes and proficiencies" with longstanding importance in mathematics education. The first of these are the NCTM process standards of problem solving, reasoning and proof, communication, representation, and connections. The second are the strands of mathematical proficiency specified in the National Research Council's report Adding It Up: adaptive reasoning, strategic competence, conceptual understanding (comprehension of mathematical concepts, operations and relations), procedural fluency (skill in carrying out procedures flexibly, accurately, efficiently and appropriately), and productive disposition (habitual inclination to see mathematics as sensible, useful, and worthwhile, coupled with a belief in diligence and one's own efficacy).

1. Make sense of problems and persevere in solving them.
Mathematically proficient students start by explaining to themselves the meaning of a problem and looking for entry points to its solution. They analyze givens, constraints, relationships, and goals. They make conjectures about the form and meaning of the solution and plan a solution pathway rather than simply jumping into a solution attempt. They consider analogous problems, and try special cases and simpler forms of the original problem in order to gain insight into its solution. They monitor and evaluate their progress and change course if necessary. Older students might, depending on the context of the problem, transform algebraic expressions or change the viewing window on their graphing calculator to get the information they need. Mathematically proficient students can explain correspondences between equations, verbal descriptions, tables, and graphs or draw diagrams of important features and relationships, graph data, and search for regularity or trends. Younger students might rely on using concrete objects or pictures to help conceptualize and solve a problem. Mathematically proficient students check their answers to problems using a different method, and they continually ask themselves, "Does this make sense?" They can understand the approaches of others to solving complex problems and identify correspondences between different approaches.

2. Reason abstractly and quantitatively.
Mathematically proficient students make sense of quantities and their relationships in problem situations. They bring two complementary abilities to bear on problems involving quantitative relationships: the ability to decontextualize—to abstract a given situation and represent it symbolically and manipulate the representing symbols as if they have a life of their own, without necessarily attending to their referents—and the ability to contextualize, to pause as needed during the manipulation process in order to probe into the referents for the symbols involved. Quantitative reasoning entails habits of creating a coherent representation of the problem at hand; considering the units involved; attending to the meaning of quantities, not just how to compute them; and knowing and flexibly using different properties of operations and objects.

3. Construct viable arguments and critique the reasoning of others.
Mathematically proficient students understand and use stated assumptions, definitions, and previously established results in constructing arguments. They make conjectures and build a logical progression of statements to explore the truth of their conjectures. They are able to analyze situations by breaking them into cases, and can recognize and use counterexamples. They justify their conclusions, communicate them to others, and respond to the arguments of others. They reason inductively about data, making plausible arguments that take into account the context from which the data arose.

Mathematically proficient students are also able to compare the effectiveness of two plausible arguments, distinguish correct logic or reasoning from that which is flawed, and—if there is a flaw in an argument—explain what it is. Elementary students can construct arguments using concrete referents such as objects, drawings, diagrams, and actions. Such arguments can make sense and be correct, even though they are not generalized or made formal until later grades. Later, students learn to determine domains to which an argument applies. Students at all grades can listen or read the arguments of others, decide whether they make sense, and ask useful questions to clarify or improve the arguments.

4. Model with mathematics.
Mathematically proficient students can apply the mathematics they know to solve problems arising in everyday life, society, and the workplace. In early grades, this might be as simple as writing an addition equation to describe a situation. In middle grades, a student might apply proportional reasoning to plan a school event or analyze a problem in the community. By high school, a student might use geometry to solve a design problem or use a function to describe how one quantity of interest depends on another. Mathematically proficient students who can apply what they know are comfortable making assumptions and approximations to simplify a complicated situation, realizing that these may need revision later. They are able to identify important quantities in a practical situation and map their relationships using such tools as diagrams, two-way tables, graphs, flowcharts and formulas. They can analyze those relationships mathematically to draw conclusions. They routinely interpret their mathematical results in the context of the situation and reflect on whether the results make sense, possibly improving the model if it has not served its purpose.

5. Use appropriate tools strategically.
Mathematically proficient students consider the available tools when solving a mathematical problem. These tools might include pencil and paper, concrete models, a ruler, a protractor, a calculator, a spreadsheet, a computer algebra system, a statistical package, or dynamic geometry software. Proficient students are sufficiently familiar with tools appropriate for their grade or course to make sound decisions about when each of these tools might be helpful, recognizing both the insight to be gained and their limitations. For example, mathematically proficient high school students analyze graphs of functions and solutions generated using a graphing calculator. They detect possible errors by strategically using estimation and other mathematical knowledge. When making mathematical models, they know that technology can enable them to visualize the results of varying assumptions, explore consequences, and compare predictions with data. Mathematically proficient students at various grade levels are able to identify relevant external mathematical resources, such as digital content located on a website, and use them to pose or solve problems. They are able to use technological tools to explore and deepen their understanding of concepts.

6. Attend to precision.
Mathematically proficient students try to communicate precisely to others. They try to use clear definitions in discussion with others and in their own reasoning. They state the meaning of the symbols they choose, including using the equal sign consistently and appropriately. They are careful about specifying units of measure, and labeling axes to clarify the correspondence with quantities in a problem. They calculate accurately and efficiently, express numerical answers with a degree of precision appropriate for the problem context. In the elementary grades, students give carefully formulated explanations to each other. By the time they reach high school they have learned to examine claims and make explicit use of definitions.

7. Look for and make use of structure.
Mathematically proficient students look closely to discern a pattern or structure. Young students, for example, might notice that three and seven more is the same amount as seven and three more, or they may sort a collection of shapes according to how many sides the shapes have. Later, students will see 7×8 *equals the well remembered* $7 \times 5 + 7 \times 3$*, in preparation for learning about the distributive property. In the expression* $x^2 + 9x + 14$*, older students can see the 14 as* 2×7 *and the 9 as* $2 + 7$*. They recognize the significance of an existing line in a geometric figure and can use the strategy of drawing an auxiliary line for solving problems. They also can step back for an overview and shift perspective. They can see complicated things, such as some algebraic expressions, as single objects or*

as being composed of several objects. For example, they can see $5 - 3(x - y)^2$ as 5 minus a positive number times a square and use that to realize that its value cannot be more than 5 for any real numbers x and y.

8. Look for and express regularity in repeated reasoning.

Mathematically proficient students notice if calculations are repeated, and look both for general methods and for shortcuts. Upper elementary students might notice when dividing 25 by 11 that they are repeating the same calculations over and over again, and conclude they have a repeating decimal. By paying attention to the calculation of slope as they repeatedly check whether points are on the line through (1, 2) with slope 3, middle school students might abstract the equation $(y - 2)/(x - 1) = 3$. Noticing the regularity in the way terms cancel when expanding $(x - 1)(x + 1)$, $(x - 1)(x^2 + x + 1)$, and $(x - 1)(x^3 + x^2 + x + 1)$ might lead them to the general formula for the sum of a geometric series. As they work to solve a problem, mathematically proficient students maintain oversight of the process, while attending to the details. They continually evaluate the reasonableness of their intermediate results.

Connecting the Standards for Mathematical Practice to the Standards for Mathematical Content

The Standards for Mathematical Practice describe ways in which developing student practitioners of the discipline of mathematics increasingly ought to engage with the subject matter as they grow in mathematical maturity and expertise throughout the elementary, middle and high school years. Designers of curricula, assessments, and professional development should all attend to the need to connect the mathematical practices to mathematical content in mathematics instruction. The Standards for Mathematical Content are a balanced combination of procedure and understanding. Expectations that begin with the word "understand" are often especially good opportunities to connect the practices to the content. Students who lack understanding of a topic may rely on procedures too heavily. Without a flexible base from which to work, they may be less likely to consider analogous problems, represent problems coherently, justify conclusions, apply the mathematics to practical situations, use technology mindfully to work with the mathematics, explain the mathematics accurately to other students, step back for an overview, or deviate from a known procedure to find a shortcut. In short, a lack of understanding effectively prevents a student from engaging in the mathematical practices. In this respect, those content standards which set an expectation of understanding are potential "points of intersection" between the Standards for Mathematical Content and the Standards for Mathematical Practice. These points of intersection are intended to be weighted toward central and generative concepts in the school mathematics curriculum that most merit the time, resources, innovative energies, and focus necessary to qualitatively improve the curriculum, instruction, assessment, professional development, and student achievement in mathematics.

Mathematics
Grade Level Expectations at a Glance

Standard	Grade Level Expectation
High School	
1. Number Sense, Properties, and Operations	1. The complex number system includes real numbers and imaginary numbers 2. Quantitative reasoning is used to make sense of quantities and their relationships in problem situations
2. Patterns, Functions, and Algebraic Structures	1. Functions model situations where one quantity determines another and can be represented algebraically, graphically, and using tables 2. Quantitative relationships in the real world can be modeled and solved using functions 3. Expressions can be represented in multiple, equivalent forms 4. Solutions to equations, inequalities and systems of equations are found using a variety of tools
3. Data Analysis, Statistics, and Probability	1. Visual displays and summary statistics condense the information in data sets into usable knowledge 2. Statistical methods take variability into account supporting informed decisions making through quantitative studies designed to answer specific questions 3. Probability models outcomes for situations in which there is inherent randomness
4. Shape, Dimension, and Geometric Relationships	1. Objects in the plane can be transformed, and those transformations can be described and analyzed mathematically 2. Concepts of similarity are foundational to geometry and its applications 3. Objects in the plane can be described and analyzed algebraically 4. Attributes of two- and three-dimensional objects are measurable and can be quantified 5. Objects in the real world can be modeled using geometric concepts

From the Common State Standards for Mathematics, Pages 58, 62, 67, 72-74, and 79.

Mathematics | High School—Number and Quantity
Numbers and Number Systems. *During the years from kindergarten to eighth grade, students must repeatedly extend their conception of number. At first, "number" means "counting number": 1, 2, 3... Soon after that, 0 is used to represent "none" and the whole numbers are formed by the counting numbers together with zero. The next extension is fractions. At first, fractions are barely numbers and tied strongly to pictorial representations. Yet by the time students understand division of fractions, they have a strong concept of fractions as numbers and have connected them, via their decimal representations, with the base-ten system used to represent the whole numbers. During middle school, fractions are augmented by negative fractions to form the rational numbers. In Grade 8, students extend this system once more, augmenting the rational numbers with the irrational numbers to form the real numbers. In high school, students will be exposed to yet another extension of number, when the real numbers are augmented by the imaginary numbers to form the complex numbers.*

With each extension of number, the meanings of addition, subtraction, multiplication, and division are extended. In each new number system—integers, rational numbers, real numbers, and complex numbers—the four operations stay the same in two important ways: They have the commutative, associative, and distributive properties and their new meanings are consistent with their previous meanings.

Extending the properties of whole-number exponents leads to new and productive notation. For example, properties of whole-number exponents suggest that $(5^{1/3})^3$ should be $5^{(1/3)3} = 5^1 = 5$ and that $5^{1/3}$ should be the cube root of 5.

Calculators, spreadsheets, and computer algebra systems can provide ways for students to become better acquainted with these new number systems and their notation. They can be used to generate data for numerical experiments, to help understand the workings of matrix, vector, and complex number algebra, and to experiment with non-integer exponents.

Quantities. *In real world problems, the answers are usually not numbers but quantities: numbers with units, which involves measurement. In their work in measurement up through Grade 8, students primarily measure commonly used attributes such as length, area, and volume. In high school, students encounter a wider variety of units in modeling, e.g., acceleration, currency conversions, derived quantities such as person-hours and heating degree days, social science rates such as per-capita income, and rates in everyday life such as points scored per game or batting averages. They also encounter novel situations in which they themselves must conceive the attributes of interest. For example, to find a good measure of overall highway safety, they might propose measures such as fatalities per year, fatalities per year per driver, or fatalities per vehicle-mile traveled. Such a conceptual process is sometimes called quantification. Quantification is important for science, as when surface area suddenly "stands out" as an important variable in evaporation. Quantification is also important for companies, which must conceptualize relevant attributes and create or choose suitable measures for them.*

Mathematics | High School—Algebra

Expressions. *An expression is a record of a computation with numbers, symbols that represent numbers, arithmetic operations, exponentiation, and, at more advanced levels, the operation of evaluating a function. Conventions about the use of parentheses and the order of operations assure that each expression is unambiguous. Creating an expression that describes a computation involving a general quantity requires the ability to express the computation in general terms, abstracting from specific instances.*

Reading an expression with comprehension involves analysis of its underlying structure. This may suggest a different but equivalent way of writing the expression that exhibits some different aspect of its meaning. For example, $p + 0.05p$ can be interpreted as the addition of a 5% tax to a price p. Rewriting $p + 0.05p$ as $1.05p$ shows that adding a tax is the same as multiplying the price by a constant factor.

Algebraic manipulations are governed by the properties of operations and exponents, and the conventions of algebraic notation. At times, an expression is the result of applying operations to simpler expressions. For example, $p + 0.05p$ is the sum of the simpler expressions p and $0.05p$. Viewing an expression as the result of operation on simpler expressions can sometimes clarify its underlying structure.

A spreadsheet or a computer algebra system (CAS) can be used to experiment with algebraic expressions, perform complicated algebraic manipulations, and understand how algebraic manipulations behave.

Equations and inequalities. *An equation is a statement of equality between two expressions, often viewed as a question asking for which values of the variables the expressions on either side are in fact equal. These values are the solutions to the equation. An identity, in contrast, is true for all values of the variables; identities are often developed by rewriting an expression in an equivalent form.*

The solutions of an equation in one variable form a set of numbers; the solutions of an equation in two variables form a set of ordered pairs of numbers, which can be plotted in the coordinate plane. Two or more equations and/or inequalities form a system. A solution for such a system must satisfy every equation and inequality in the system.

An equation can often be solved by successively deducing from it one or more simpler equations. For example, one can add the same constant to both sides without changing the solutions, but squaring both sides might lead to extraneous solutions. Strategic competence in solving includes looking ahead for productive manipulations and anticipating the nature and number of solutions.

Some equations have no solutions in a given number system, but have a solution in a larger system. For example, the solution of $x + 1 = 0$ is an integer, not a whole number; the solution of $2x + 1 = 0$ is a rational number, not an integer; the solutions of $x^2 - 2 = 0$ are real numbers, not rational numbers; and the solutions of $x^2 + 2 = 0$ are complex numbers, not real numbers.

The same solution techniques used to solve equations can be used to rearrange formulas. For example, the formula for the area of a trapezoid, $A = ((b_1+b_2)/2)h$, can be solved for h using the same deductive process.

Inequalities can be solved by reasoning about the properties of inequality. Many, but not all, of the properties of equality continue to hold for inequalities and can be useful in solving them.

Connections to Functions and Modeling. *Expressions can define functions, and equivalent expressions define the same function. Asking when two functions have the same value for the same input leads to an equation; graphing the two functions allows for finding approximate solutions of the equation. Converting a verbal description to an equation, inequality, or system of these is an essential skill in modeling.*

Mathematics | High School—Functions

Functions describe situations where one quantity determines another. For example, the return on $10,000 invested at an annualized percentage rate of 4.25% is a function of the length of time the money is invested. Because we continually make theories about dependencies between quantities in nature and society, functions are important tools in the construction of mathematical models.

In school mathematics, functions usually have numerical inputs and outputs and are often defined by an algebraic expression. For example, the time in hours it takes for a car to drive 100 miles is a function of the car's speed in miles per hour, v; the rule $T(v) = 100/v$ expresses this relationship algebraically and defines a function whose name is T.

The set of inputs to a function is called its domain. We often infer the domain to be all inputs for which the expression defining a function has a value, or for which the function makes sense in a given context.

A function can be described in various ways, such as by a graph (e.g., the trace of a seismograph); by a verbal rule, as in, "I'll give you a state, you give me the capital city;" by an algebraic expression like $f(x) = a + bx$; or by a recursive rule. The graph of a function is often a useful way of visualizing the relationship of the function models, and manipulating a mathematical expression for a function can throw light on the function's properties.

Functions presented as expressions can model many important phenomena. Two important families of functions characterized by laws of growth are linear functions, which grow at a constant rate, and exponential functions, which grow at a constant percent rate. Linear functions with a constant term of zero describe proportional relationships.

A graphing utility or a computer algebra system can be used to experiment with properties of these functions and their graphs and to build computational models of functions, including recursively defined functions.

Connections to Expressions, Equations, Modeling, and Coordinates.

Determining an output value for a particular input involves evaluating an expression; finding inputs that yield a given output involves solving an equation. Questions about when two functions have the *same value for the same input lead to equations, whose solutions can be visualized from the intersection of their graphs. Because functions describe relationships between quantities, they are frequently used in modeling. Sometimes functions are defined by a recursive process, which can be displayed effectively using a spreadsheet or other technology.*

Mathematics | High School—Modeling

Modeling links classroom mathematics and statistics to everyday life, work, and decision-making. Modeling is the process of choosing and using appropriate mathematics and statistics to analyze empirical situations, to understand them better, and to improve decisions. Quantities and their relationships in physical, economic, public policy, social, and everyday situations can be modeled using mathematical and statistical methods. When making mathematical models, technology is valuable for varying assumptions, exploring consequences, and comparing predictions with data.

A model can be very simple, such as writing total cost as a product of unit price and number bought, or using a geometric shape to describe a physical object like a coin. Even such simple models involve making choices. It is up to us whether to model a coin as a three-dimensional cylinder, or whether a two-dimensional disk works well enough for our purposes. Other situations—modeling a delivery route, a production schedule, or a comparison of loan amortizations—need more elaborate models that use other tools from the mathematical sciences. Real-world situations are not organized and labeled for analysis; formulating tractable models, representing such models, and analyzing them is appropriately a creative process. Like every such process, this depends on acquired expertise as well as creativity.

Some examples of such situations might include:

- *Estimating how much water and food is needed for emergency relief in a devastated city of 3 million people, and how it might be distributed.*
- *Planning a table tennis tournament for 7 players at a club with 4 tables, where each player plays against each other player.*
- *Designing the layout of the stalls in a school fair so as to raise as much money as possible.*
- *Analyzing stopping distance for a car.*
- *Modeling savings account balance, bacterial colony growth, or investment growth.*
- *Engaging in critical path analysis, e.g., applied to turnaround of an aircraft at an airport.*
- *Analyzing risk in situations such as extreme sports, pandemics, and terrorism.*
- *Relating population statistics to individual predictions.*

In situations like these, the models devised depend on a number of factors: How precise an answer do we want or need? What aspects of the situation do we most need to understand, control, or optimize? What resources of time and tools do we have? The range of models that we can create and analyze is also constrained by the limitations of our mathematical, statistical, and technical skills, and our ability to recognize significant variables and relationships among them. Diagrams of various kinds, spreadsheets and other technology, and algebra are powerful tools for understanding and solving problems drawn from different types of real-world situations.

One of the insights provided by mathematical modeling is that essentially the same mathematical or statistical structure can sometimes model seemingly different situations. Models can also shed light on the mathematical structures themselves, for example, as when a model of bacterial growth makes more vivid the explosive growth of the exponential function.

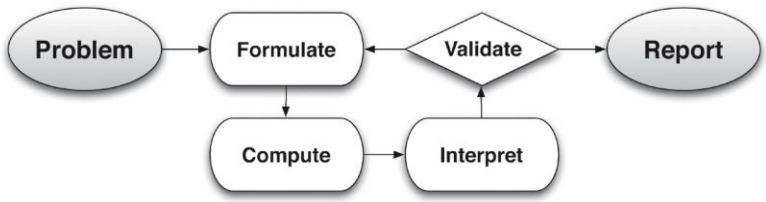
The basic modeling cycle is summarized in the diagram (below). It involves (1) identifying variables in the situation and selecting those that represent essential features, (2) formulating a model by creating and selecting geometric, graphical, tabular, algebraic, or statistical representations that describe relationships between the variables, (3) analyzing and performing operations on these relationships to draw conclusions, (4) interpreting the results of the mathematics in terms of the original situation, (5) validating the conclusions by comparing them with the situation, and then either improving the model or, if it is acceptable, (6) reporting on the conclusions and the reasoning behind them. Choices, assumptions, and approximations are present throughout this cycle.

In descriptive modeling, a model simply describes the phenomena or summarizes them in a compact form. Graphs of observations are a familiar descriptive model— for example, graphs of global temperature and atmospheric CO2 over time.

Analytic modeling seeks to explain data on the basis of deeper theoretical ideas, albeit with parameters that are empirically based; for example, exponential growth of bacterial colonies (until cut-off mechanisms such as pollution or starvation intervene) follows from a constant reproduction rate. Functions are an important tool for analyzing such problems.

Graphing utilities, spreadsheets, computer algebra systems, and dynamic geometry software are powerful tools that can be used to model purely mathematical phenomena (e.g., the behavior of polynomials) as well as physical phenomena.

***Modeling Standards**. Modeling is best interpreted not as a collection of isolated topics but rather in relation to other standards. Making mathematical models is a Standard for Mathematical Practice, and specific modeling standards appear throughout the high school standards indicated by a star symbol (*).*



Mathematics | High School—Geometry

An understanding of the attributes and relationships of geometric objects can be applied in diverse contexts—interpreting a schematic drawing, estimating the amount of wood needed to frame a sloping roof, rendering computer graphics, or designing a sewing pattern for the most efficient use of material.

Although there are many types of geometry, school mathematics is devoted primarily to plane Euclidean geometry, studied both synthetically (without coordinates) and analytically (with coordinates). Euclidean geometry is characterized most importantly by the Parallel Postulate, that through a point not on a given line there is exactly one parallel line. (Spherical geometry, in contrast, has no parallel lines.)

During high school, students begin to formalize their geometry experiences from elementary and middle school, using more precise definitions and developing careful proofs. Later in college some students develop Euclidean and other geometries carefully from a small set of axioms.

The concepts of congruence, similarity, and symmetry can be understood from the perspective of geometric transformation. Fundamental are the rigid motions: translations, rotations, reflections, and combinations of these, all of which are here assumed to preserve distance and angles (and therefore shapes generally). Reflections and rotations each explain a particular type of symmetry, and the symmetries of an object offer insight into its attributes—as when the reflective symmetry of an isosceles triangle assures that its base angles are congruent.

In the approach taken here, two geometric figures are defined to be congruent if there is a sequence of rigid motions that carries one onto the other. This is the principle of superposition. For triangles, congruence means the equality of all corresponding pairs of sides and all corresponding pairs of angles. During the middle grades, through experiences drawing triangles from given conditions, students notice ways to specify enough measures in a triangle to ensure that all triangles drawn with those measures are congruent. Once these triangle congruence criteria (ASA, SAS, and SSS) are established using rigid motions, they can be used to prove theorems about triangles, quadrilaterals, and other geometric figures.

Similarity transformations (rigid motions followed by dilations) define similarity in the same way that rigid motions define congruence, thereby formalizing the similarity ideas of "same shape" and "scale factor" developed in the middle grades. These transformations lead to the criterion for triangle similarity that two pairs of corresponding angles are congruent.

The definitions of sine, cosine, and tangent for acute angles are founded on right triangles and similarity, and, with the Pythagorean Theorem, are fundamental in many real-world and theoretical situations. The Pythagorean Theorem is generalized to nonright triangles by the Law of Cosines. Together, the Laws of Sines and Cosines embody the triangle congruence criteria for the cases where three pieces of information suffice to completely solve a triangle. Furthermore, these laws yield two possible solutions in the ambiguous case, illustrating that Side-Side-Angle is not a congruence criterion.

Analytic geometry connects algebra and geometry, resulting in powerful methods of analysis and problem solving. Just as the number line associates numbers with locations in one dimension, a pair of perpendicular axes associates pairs of numbers with locations in two dimensions. This correspondence between numerical coordinates and geometric points allows methods from algebra to be applied to geometry and vice versa. The solution set of an equation becomes a geometric curve, making visualization a tool for doing and understanding algebra. Geometric shapes can be described by equations, making algebraic manipulation into a tool for geometric understanding, modeling, and proof. Geometric transformations of the graphs of equations correspond to algebraic changes in their equations.

Dynamic geometry environments provide students with experimental and modeling tools that allow them to investigate geometric phenomena in much the same way as computer algebra systems allow them to experiment with algebraic phenomena.

Connections to Equations. *The correspondence between numerical coordinates and geometric points allows methods from algebra to be applied to geometry and vice versa. The solution set of an equation becomes a geometric curve, making visualization a tool for doing and understanding algebra. Geometric shapes can be described by equations, making algebraic manipulation into a tool for geometric understanding, modeling, and proof.*

Mathematics | High School—Statistics and Probability*

Decisions or predictions are often based on data—numbers in context. These decisions or predictions would be easy if the data always sent a clear message, but the message is often obscured by variability. Statistics provides tools for describing variability in data and for making informed decisions that take it into account.

Data are gathered, displayed, summarized, examined, and interpreted to discover patterns and deviations from patterns. Quantitative data can be described in terms of key characteristics: measures of shape, center, and spread. The shape of a data distribution might be described as symmetric, skewed, flat, or bell shaped, and it might be summarized by a statistic measuring center (such as mean or median) and a statistic measuring spread (such as standard deviation or interquartile range). Different distributions can be compared numerically using these statistics or compared visually using plots. Knowledge of center and spread are not enough to describe a distribution. Which statistics to compare, which plots to use, and what the results of a comparison might mean, depend on the question to be investigated and the real-life actions to be taken.

Randomization has two important uses in drawing statistical conclusions. First, collecting data from a random sample of a population makes it possible to draw valid conclusions about the whole population, taking variability into account. Second, randomly assigning individuals to different treatments allows a fair comparison of the effectiveness of those treatments. A statistically significant outcome is one that is unlikely to be due to chance alone, and this can be evaluated only under the condition of randomness. The conditions under which data are collected are important in drawing conclusions from the data; in critically reviewing uses of statistics in public media and other reports, it is important to *consider the study design, how the data were gathered, and the analyses employed as well as the data summaries and the conclusions drawn.*

Random processes can be described mathematically by using a probability model: a list or description of the possible outcomes (the sample space), each of which is assigned a probability. In situations such as flipping a coin, rolling a number cube, or drawing a card, it might be reasonable to assume various outcomes are equally likely. In a probability model, sample points represent outcomes and combine to make up events; probabilities of events can be computed by applying the Addition and Multiplication Rules. Interpreting these probabilities relies on an understanding of independence and conditional probability, which can be approached through the analysis of two-way tables.

Technology plays an important role in statistics and probability by making it possible to generate plots, regression functions, and correlation coefficients, and to simulate many possible outcomes in a short amount of time.

***Connections to Functions and Modeling**. Functions may be used to describe data; if the data suggest a linear relationship, the relationship can be modeled with a regression line, and its strength and direction can be expressed through a correlation coefficient.*

21^{st} Century Skills and Readiness Competencies in Mathematics

Mathematics in Colorado's description of 21^{st} century skills is a synthesis of the essential abilities students must apply in our rapidly changing world. Today's mathematics students need a repertoire of knowledge and skills that are more diverse, complex, and integrated than any previous generation. Mathematics is inherently demonstrated in each of Colorado 21^{st} century skills, as follows:

Critical Thinking and Reasoning
Mathematics is a discipline grounded in critical thinking and reasoning. Doing mathematics involves recognizing problematic aspects of situations, devising and carrying out strategies, evaluating the reasonableness of solutions, and justifying methods, strategies, and solutions. Mathematics provides the grammar and structure that make it possible to describe patterns that exist in nature and society.

Information Literacy
The discipline of mathematics equips students with tools and habits of mind to organize and interpret quantitative data. Informationally literate mathematics students effectively use learning tools, including technology, and clearly communicate using mathematical language.

Collaboration
Mathematics is a social discipline involving the exchange of ideas. In the course of doing mathematics, students offer ideas, strategies, solutions, justifications, and proofs for others to evaluate. In turn, the mathematics student interprets and evaluates the ideas, strategies, solutions, justifications and proofs of others.

Self-Direction
Doing mathematics requires a productive disposition and self-direction. It involves monitoring and assessing one's mathematical thinking and persistence in searching for patterns, relationships, and sensible solutions.

Invention
Mathematics is a dynamic discipline, ever expanding as new ideas are contributed. Invention is the key element as students make and test conjectures, create mathematical models of real-world phenomena, generalize results, and make connections among ideas, strategies and solutions.

Colorado's Description for School Readiness
(Adopted by the State Board of Education, December 2008)
School readiness describes both the preparedness of a child to engage in and benefit from learning experiences, and the ability of a school to meet the needs of all students enrolled in publicly funded preschools or kindergartens. School readiness is enhanced when schools, families, and community service providers work collaboratively to ensure that every child is ready for higher levels of learning in academic content.

Colorado's Description of Postsecondary and Workforce Readiness
(Adopted by the State Board of Education, June 2009)
Postsecondary and workforce readiness describes the knowledge, skills, and behaviors essential for high school graduates to be prepared to enter college and the workforce and to compete in the global economy. The description assumes students have developed consistent intellectual growth throughout their high school career as a result of academic work that is increasingly challenging, engaging, and coherent. Postsecondary education and workforce readiness assumes that students are ready and able to demonstrate the following without the need for remediation: Critical thinking and problem-solving; finding and using information/information technology; creativity and innovation; global and cultural awareness; civic responsibility; work ethic; personal responsibility; communication; and collaboration.

How These Skills and Competencies are Embedded in the Revised Standards
Three themes are used to describe these important skills and competencies and are interwoven throughout the standards: *inquiry questions; relevance and application; and the nature of each discipline.* These competencies should not be thought of stand-alone concepts, but should be integrated throughout the curriculum in all grade levels. Just as it is impossible to teach thinking skills to students without the content to think about, it is equally impossible for students to understand the content of a discipline without grappling with complex questions and the investigation of topics.

Inquiry Questions – Inquiry is a multifaceted process requiring students to think and pursue understanding. Inquiry demands that students (a) engage in an active observation and questioning process; (b) investigate to gather evidence; (c) formulate explanations based on evidence; (d) communicate and justify explanations, and; (e) reflect and refine ideas. Inquiry is more than hands-on activities; it requires students to cognitively wrestle with core concepts as they make sense of new ideas.

Relevance and Application – The hallmark of learning a discipline is the ability to apply the knowledge, skills, and concepts in real-world, relevant contexts. Components of this include solving problems, developing, adapting, and refining solutions for the betterment of society. The application of a discipline, including how technology assists or accelerates the work, enables students to more fully appreciate how the mastery of the grade level expectation matters after formal schooling is complete.

Nature of Discipline – The unique advantage of a discipline is the perspective it gives the mind to see the world and situations differently. The characteristics and viewpoint one keeps as a result of mastering the grade level expectation is the nature of the discipline retained in the mind's eye.

1. Number Sense, Properties, and Operations

Number sense provides students with a firm foundation in mathematics. Students build a deep understanding of quantity, ways of representing numbers, relationships among numbers, and number systems. Students learn that numbers are governed by properties, and understanding these properties leads to fluency with operations.

Prepared Graduates
The prepared graduate competencies are the preschool through twelfth-grade concepts and skills that all students who complete the Colorado education system must master to ensure their success in a postsecondary and workforce setting.

Prepared Graduate Competencies in the Number Sense, Properties, and Operations Standard are:

- Understand the structure and properties of our number system. At their most basic level numbers are abstract symbols that represent real-world quantities
- Understand quantity through estimation, precision, order of magnitude, and comparison. The reasonableness of answers relies on the ability to judge appropriateness, compare, estimate, and analyze error
- Are fluent with basic numerical and symbolic facts and algorithms, and are able to select and use appropriate (mental math, paper and pencil, and technology) methods based on an understanding of their efficiency, precision, and transparency
- Make both relative (multiplicative) and absolute (arithmetic) comparisons between quantities. Multiplicative thinking underlies proportional reasoning
- Understand that equivalence is a foundation of mathematics represented in numbers, shapes, measures, expressions, and equations
- Apply transformation to numbers, shapes, functional representations, and data

Content Area: Mathematics
Standard: 1. Number Sense, Properties, and Operations

Prepared Graduates:

- Understand the structure and properties of our number system. At their most basic level numbers are abstract symbols that represent real-world quantities

Grade Level Expectation: High School

Concepts and skills students master:

1. The complex number system includes real numbers and imaginary numbers

Evidence Outcomes	21st Century Skills and Readiness Competencies
Students can: a. Extend the properties of exponents to rational exponents. (CCSS: N-RN) i. Explain how the definition of the meaning of rational exponents follows from extending the properties of integer exponents to those values, allowing for a notation for radicals in terms of rational exponents.[1] (CCSS: N-RN.1) ii. Rewrite expressions involving radicals and rational exponents using the properties of exponents. (CCSS: N-RN.2) b. Use properties of rational and irrational numbers. (CCSS: N-RN) i. Explain why the sum or product of two rational numbers is rational. (CCSS: N-RN.3) ii. Explain why the sum of a rational number and an irrational number is irrational. (CCSS: N-RN.3) iii. Explain why the product of a nonzero rational number and an irrational number is irrational. (CCSS: N-RN.3) c. Perform arithmetic operations with complex numbers. (CCSS: N-CN) i. Define the complex number i such that $i^2 = -1$, and show that every complex number has the form $a + bi$ where a and b are real numbers. (CCSS: N-CN.1) ii. Use the relation $i^2 = -1$ and the commutative, associative, and distributive properties to add, subtract, and multiply complex numbers. (CCSS: N-CN.2) d. Use complex numbers in polynomial identities and equations. (CCSS: N-CN) i. Solve quadratic equations with real coefficients that have complex solutions. (CCSS: N-CN.7)	**Inquiry Questions:** 1. When you extend to a new number systems (e.g., from integers to rational numbers and from rational numbers to real numbers), what properties apply to the extended number system? 2. Are there more complex numbers than real numbers? 3. What is a number system? 4. Why are complex numbers important? **Relevance and Application:** 1. Complex numbers have applications in fields such as chaos theory and fractals. The familiar image of the Mandelbrot fractal is the Mandelbrot set graphed on the complex plane. **Nature of Mathematics:** 1. Mathematicians build a deep understanding of quantity, ways of representing numbers, and relationships among numbers and number systems. 2. Mathematics involves making and testing conjectures, generalizing results, and making connections among ideas, strategies, and solutions. 3. Mathematicians look for and make use of structure. (MP) 4. Mathematicians look for and express regularity in repeated reasoning. (MP)

Content Area: Mathematics
Standard: 1. Number Sense, Properties, and Operations

Prepared Graduates:

- Understand quantity through estimation, precision, order of magnitude, and comparison. The reasonableness of answers relies on the ability to judge appropriateness, compare, estimate, and analyze error

Grade Level Expectation: High School

Concepts and skills students master:

2. Quantitative reasoning is used to make sense of quantities and their relationships in problem situations

Evidence Outcomes	21st Century Skills and Readiness Competencies
Students can: a. Reason quantitatively and use units to solve problems (CCSS: N-Q) i. Use units as a way to understand problems and to guide the solution of multi-step problems. (CCSS: N-Q.1) 1. Choose and interpret units consistently in formulas. (CCSS: N-Q.1) 2. Choose and interpret the scale and the origin in graphs and data displays. (CCSS: N-Q.1) ii. Define appropriate quantities for the purpose of descriptive modeling. (CCSS: N-Q.2) iii. Choose a level of accuracy appropriate to limitations on measurement when reporting quantities. (CCSS: N-Q.3) iv. Describe factors affecting take-home pay and calculate the impact (PFL) v. Design and use a budget, including income (net take-home pay) and expenses (mortgage, car loans, and living expenses) to demonstrate how living within your means is essential for a secure financial future (PFL)	**Inquiry Questions:** 1. Can numbers ever be too big or too small to be useful? 2. How much money is enough for retirement? (PFL) 3. What is the return on investment of post-secondary educational opportunities? (PFL) **Relevance and Application:** 1. The choice of the appropriate measurement tool meets the precision requirements of the measurement task. For example, using a caliper for the manufacture of brake discs or a tape measure for pant size. 2. The reading, interpreting, and writing of numbers in scientific notation with and without technology is used extensively in the natural sciences such as representing large or small quantities such as speed of light, distance to other planets, distance between stars, the diameter of a cell, and size of a micro–organism. 3. Fluency with computation and estimation allows individuals to analyze aspects of personal finance, such as calculating a monthly budget, estimating the amount left in a checking account, making informed purchase decisions, and computing a probable paycheck given a wage (or salary), tax tables, and other deduction schedules. **Nature of Mathematics:** 1. Using mathematics to solve a problem requires choosing what mathematics to use; making simplifying assumptions, estimates, or approximations; computing; and checking to see whether the solution makes sense. 2. Mathematicians reason abstractly and quantitatively. (MP) 3. Mathematicians attend to precision. (MP)

[1] For example, we define $5^{1/3}$ to be the cube root of 5 because we want $(5^{1/3})^3 = 5^{(1/3)3}$ to hold, so $(5^{1/3})^3$ must equal 5. (CCSS: N-RN.1)

2. Patterns, Functions, and Algebraic Structures

Pattern sense gives students a lens with which to understand trends and commonalities. Being a student of mathematics involves recognizing and representing mathematical relationships and analyzing change. Students learn that the structures of algebra allow complex ideas to be expressed succinctly.

Prepared Graduates
The prepared graduate competencies are the preschool through twelfth-grade concepts and skills that all students who complete the Colorado education system must have to ensure success in a postsecondary and workforce setting.

Prepared Graduate Competencies in the 2. Patterns, Functions, and Algebraic Structures Standard are:

- Are fluent with basic numerical and symbolic facts and algorithms, and are able to select and use appropriate (mental math, paper and pencil, and technology) methods based on an understanding of their efficiency, precision, and transparency
- Understand that equivalence is a foundation of mathematics represented in numbers, shapes, measures, expressions, and equations
- Make sound predictions and generalizations based on patterns and relationships that arise from numbers, shapes, symbols, and data
- Make claims about relationships among numbers, shapes, symbols, and data and defend those claims by relying on the properties that are the structure of mathematics
- Use critical thinking to recognize problematic aspects of situations, create mathematical models, and present and defend solutions

Content Area: Mathematics
Standard: 2. Patterns, Functions, and Algebraic Structures

Prepared Graduates:

- Make sound predictions and generalizations based on patterns and relationships that arise from numbers, shapes, symbols, and data

Grade Level Expectation: High School

Concepts and skills students master:

1. Functions model situations where one quantity determines another and can be represented algebraically, graphically, and using tables

Evidence Outcomes	21st Century Skills and Readiness Competencies
Students can: a. Formulate the concept of a function and use function notation. (CCSS: F-IF) i. Explain that a function is a correspondence from one set (called the domain) to another set (called the range) that assigns to each element of the domain exactly one element of the range.[1] (CCSS: F-IF.1) ii. Use function notation, evaluate functions for inputs in their domains, and interpret statements that use function notation in terms of a context. (CCSS: F-IF.2) iii. Demonstrate that sequences are functions,[2] sometimes defined recursively, whose domain is a subset of the integers. (CCSS: F-IF.3) b. Interpret functions that arise in applications in terms of the context. (CCSS: F-IF) i. For a function that models a relationship between two quantities, interpret key features of graphs and tables in terms of the quantities, and sketch graphs showing key features[3] given a verbal description of the relationship. ★ (CCSS: F-IF.4) ii. Relate the domain of a function to its graph and, where applicable, to the quantitative relationship it describes.[4] ★ (CCSS: F-IF.5) iii. Calculate and interpret the average rate of change[5] of a function over a specified interval. Estimate the rate of change from a graph.★ (CCSS: F-IF.6) c. Analyze functions using different representations. (CCSS: F-IF) i. Graph functions expressed symbolically and show key features of the graph, by hand in simple cases and using technology for more complicated cases. ★ (CCSS: F-IF.7) ii. Graph linear and quadratic functions and show intercepts, maxima, and minima. (CCSS: F-IF.7a) iii. Graph square root, cube root, and piecewise-defined functions, including step functions and absolute value functions. (CCSS: F-IF.7b) iv. Graph polynomial functions, identifying zeros when suitable factorizations are available, and showing end behavior. (CCSS: F-IF.7c) v. Graph exponential and logarithmic functions, showing intercepts and end behavior, and trigonometric functions, showing period, midline, and amplitude. (CCSS: F-IF.7e) vi. Write a function defined by an expression in different but equivalent forms to reveal and explain different properties of the function. (CCSS: F-IF.8)	**Inquiry Questions:** 1. Why are relations and functions represented in multiple ways? 2. How can a table, graph, and function notation be used to explain how one function family is different from and/or similar to another? 3. What is an inverse? 4. How is "inverse function" most likely related to addition and subtraction being inverse operations and to multiplication and division being inverse operations? 5. How are patterns and functions similar and different? 6. How could you visualize a function with four variables, such as $x^2+y^2+z^2+w^2=1$? 7. Why couldn't people build skyscrapers without using functions? 8. How do symbolic transformations affect an equation, inequality, or expression? **Relevance and Application:** 1. Knowledge of how to interpret rate of change of a function allows investigation of rate of return and time on the value of investments. (PFL) 2. Comprehension of rate of change of a function is important preparation for the study of calculus. 3. The ability to analyze a function for the intercepts, asymptotes, domain, range, and local and global behavior provides insights into the situations modeled by the function. For example, epidemiologists could compare the rate of flu infection among people who received flu shots to the rate of flu infection among people who did not receive a flu shot to gain insight into the effectiveness of the flu shot. 4. The exploration of multiple representations of functions develops a deeper understanding of the relationship

1. Use the process of factoring and completing the square in a quadratic function to show zeros, extreme values, and symmetry of the graph, and interpret these in terms of a context. (CCSS: F-IF.8a) 2. Use the properties of exponents to interpret expressions for exponential functions.[6] (CCSS: F-IF.8b) 3. Compare properties of two functions each represented in a different way[7] (algebraically, graphically, numerically in tables, or by verbal descriptions). (CCSS: F-IF.9) d. Build a function that models a relationship between two quantities. (CCSS: F-BF) i. Write a function that describes a relationship between two quantities.★ (CCSS: F-BF.1) 1. Determine an explicit expression, a recursive process, or steps for calculation from a context. (CCSS: F-BF.1a) 2. Combine standard function types using arithmetic operations.[8] (CCSS: F-BF.1b) ii. Write arithmetic and geometric sequences both recursively and with an explicit formula, use them to model situations, and translate between the two forms.★ (CCSS: F-BF.2) e. Build new functions from existing functions. (CCSS: F-BF) i. Identify the effect on the graph of replacing $f(x)$ by $f(x) + k$, $k\,f(x)$, $f(kx)$, and $f(x + k)$ for specific values of *k,*[9] and find the value of *k* given the graphs.[10] (CCSS: F-BF.3) ii. Experiment with cases and illustrate an explanation of the effects on the graph using technology. iii. Find inverse functions.[11] (CCSS: F-BF.4) f. Extend the domain of trigonometric functions using the unit circle. (CCSS: F-TF) i. Use radian measure of an angle as the length of the arc on the unit circle subtended by the angle. (CCSS: F-TF.1) ii. Explain how the unit circle in the coordinate plane enables the extension of trigonometric functions to all real numbers, interpreted as radian measures of angles traversed counterclockwise around the unit circle. (CCSS: F-TF.2) **Indicates a part of the standard connected to the mathematical practice of Modeling*	between the variables in the function. 5. The understanding of the relationship between variables in a function allows people to use functions to model relationships in the real world such as compound interest, population growth and decay, projectile motion, or payment plans. 6. Comprehension of slope, intercepts, and common forms of linear equations allows easy retrieval of information from linear models such as rate of growth or decrease, an initial charge for services, speed of an object, or the beginning balance of an account. 7. Understanding sequences is important preparation for calculus. Sequences can be used to represent functions including e^x, e^{x^2}, $\sin x$, and $\cos x$. **Nature of Mathematics:** 1. Mathematicians use multiple representations of functions to explore the properties of functions and the properties of families of functions. 2. Mathematicians model with mathematics. (MP) 3. Mathematicians use appropriate tools strategically. (MP) 4. Mathematicians look for and make use of structure. (MP)

Content Area: Mathematics
Standard: 2. Patterns, Functions, and Algebraic Structures

Prepared Graduates:

- Use critical thinking to recognize problematic aspects of situations, create mathematical models, and present and defend solutions

Grade Level Expectation: High School

Concepts and skills students master:

2. Quantitative relationships in the real world can be modeled and solved using functions

Evidence Outcomes

Students can:

a. Construct and compare linear, quadratic, and exponential models and solve problems. (CCSS: F-LE)
   i. Distinguish between situations that can be modeled with linear functions and with exponential functions. (CCSS: F-LE.1)
      1. Prove that linear functions grow by equal differences over equal intervals, and that exponential functions grow by equal factors over equal intervals. (CCSS: F-LE.1a)
      2. Identify situations in which one quantity changes at a constant rate per unit interval relative to another. (CCSS: F-LE.1b)
      3. Identify situations in which a quantity grows or decays by a constant percent rate per unit interval relative to another. (CCSS: F-LE.1c)
   ii. Construct linear and exponential functions, including arithmetic and geometric sequences, given a graph, a description of a relationship, or two input-output pairs.[12] (CCSS: F-LE.2)
   iii. Use graphs and tables to describe that a quantity increasing exponentially eventually exceeds a quantity increasing linearly, quadratically, or (more generally) as a polynomial function. (CCSS: F-LE.3)
   iv. For exponential models, express as a logarithm the solution to $ab^{ct} = d$ where a, c, and d are numbers and the base b is 2, 10, or e; evaluate the logarithm using technology. (CCSS: F-LE.4)

b. Interpret expressions for function in terms of the situation they model. (CCSS: F-LE)
   i. Interpret the parameters in a linear or exponential function in terms of a context. (CCSS: F-LE.5)

c. Model periodic phenomena with trigonometric functions. (CCSS: F-TF)
   i. Choose the trigonometric functions to model periodic phenomena with specified amplitude, frequency, and midline. ★ (CCSS: F-TF.5)

d. Model personal financial situations
   i. Analyze* the impact of interest rates on a personal financial plan (PFL)
   ii. Evaluate* the costs and benefits of credit (PFL)
   iii. Analyze various lending sources, services, and financial institutions (PFL)

**Indicates a part of the standard connected to the mathematical practice of Modeling.*

21st Century Skills and Readiness Competencies

Inquiry Questions:

1. Why do we classify functions?
2. What phenomena can be modeled with particular functions?
3. Which financial applications can be modeled with exponential functions? Linear functions? (PFL)
4. What elementary function or functions best represent a given scatter plot of two-variable data?
5. How much would today's purchase cost tomorrow? (PFL)

Relevance and Application:

1. The understanding of the qualitative behavior of functions allows interpretation of the qualitative behavior of systems modeled by functions such as time-distance, population growth, decay, heat transfer, and temperature of the ocean versus depth.
2. The knowledge of how functions model real-world phenomena allows exploration and improved understanding of complex systems such as how population growth may affect the environment , how interest rates or inflation affect a personal budget, how stopping distance is related to reaction time and velocity, and how volume and temperature of a gas are related.
3. Biologists use polynomial curves to model the shapes of jaw bone fossils. They analyze the polynomials to find potential evolutionary relationships among the species.
4. Physicists use basic linear and quadratic functions to model the motion of projectiles.

Nature of Mathematics:

1. Mathematicians use their knowledge of functions to create accurate models of complex systems.
2. Mathematicians use models to better understand systems and make predictions about future systemic behavior.
3. Mathematicians reason abstractly and quantitatively. (MP)
4. Mathematicians construct viable arguments and critique the reasoning of others. (MP)
5. Mathematicians model with mathematics. (MP)

Content Area: Mathematics
Standard: 2. Patterns, Functions, and Algebraic Structures

Prepared Graduates:

- Understand that equivalence is a foundation of mathematics represented in numbers, shapes, measures, expressions, and equations

Grade Level Expectation: High School

Concepts and skills students master:

3. Expressions can be represented in multiple, equivalent forms

Evidence Outcomes	21st Century Skills and Readiness Competencies
Students can: a. Interpret the structure of expressions.(CCSS: A-SSE) i. Interpret expressions that represent a quantity in terms of its context.★ (CCSS: A-SSE.1) 1. Interpret parts of an expression, such as terms, factors, and coefficients. (CCSS: A-SSE.1a) 2. Interpret complicated expressions by viewing one or more of their parts as a single entity.[13] (CCSS: A-SSE.1b) ii. Use the structure of an expression to identify ways to rewrite it.[14] (CCSS: A-SSE.2) b. Write expressions in equivalent forms to solve problems. (CCSS: A-SSE) i. Choose and produce an equivalent form of an expression to reveal and explain properties of the quantity represented by the expression.★ (CCSS: A-SSE.3) 1. Factor a quadratic expression to reveal the zeros of the function it defines. (CCSS: A-SSE.3a) 2. Complete the square in a quadratic expression to reveal the maximum or minimum value of the function it defines. (CCSS: A-SSE.3b) 3. Use the properties of exponents to transform expressions for exponential functions.[15] (CCSS: A-SSE.3c) ii. Derive the formula for the sum of a finite geometric series (when the common ratio is not 1), and use the formula to solve problems.[16]★ (CCSS: A-SSE.4) c. Perform arithmetic operations on polynomials. (CCSS: A-APR) i. Explain that polynomials form a system analogous to the integers, namely, they are closed under the operations of addition, subtraction, and multiplication; add, subtract, and multiply polynomials. (CCSS: A-APR.1) d. Understand the relationship between zeros and factors of polynomials. (CCSS: A-APR) i. State and apply the Remainder Theorem.[17] (CCSS: A-APR.2) ii. Identify zeros of polynomials when suitable factorizations are available, and use the zeros to construct a rough graph of the function defined by the polynomial. (CCSS: A-APR.3) e. Use polynomial identities to solve problems. (CCSS: A-APR) i. Prove polynomial identities[18] and use them to describe numerical relationships. (CCSS: A-APR.4) f. Rewrite rational expressions. (CCSS: A-APR) g. Rewrite simple rational expressions in different forms.[19] (CCSS: A-APR.6) **Indicates a part of the standard connected to the mathematical practice of Modeling*	**Inquiry Questions:** 1. When is it appropriate to simplify expressions? 2. The ancient Greeks multiplied binomials and found the roots of quadratic equations without algebraic notation. How can this be done? **Relevance and Application:** 1. The simplification of algebraic expressions and solving equations are tools used to solve problems in science. Scientists represent relationships between variables by developing a formula and using values obtained from experimental measurements and algebraic manipulation to determine values of quantities that are difficult or impossible to measure directly such as acceleration due to gravity, speed of light, and mass of the earth. 2. The manipulation of expressions and solving formulas are techniques used to solve problems in geometry such as finding the area of a circle, determining the volume of a sphere, calculating the surface area of a prism, and applying the Pythagorean Theorem. **Nature of Mathematics:** 1. Mathematicians abstract a problem by representing it as an equation. They travel between the concrete problem and the abstraction to gain insights and find solutions. 2. Mathematicians construct viable arguments and critique the reasoning of others. (MP) 3. Mathematicians model with mathematics. (MP) 4. Mathematicians look for and express regularity in repeated reasoning. (MP)

Content Area: Mathematics
Standard: 2. Patterns, Functions, and Algebraic Structures

Prepared Graduates:

- Are fluent with basic numerical and symbolic facts and algorithms, and are able to select and use appropriate (mental math, paper and pencil, and technology) methods based on an understanding of their efficiency, precision, and transparency

Grade Level Expectation: High School

Concepts and skills students master:

4. Solutions to equations, inequalities and systems of equations are found using a variety of tools

Evidence Outcomes

Students can:

a. Create equations that describe numbers or relationships. (CCSS: A-CED)
   i. Create equations and inequalities[20] in one variable and use them to solve problems. (CCSS: A-CED.1)
   ii. Create equations in two or more variables to represent relationships between quantities and graph equations on coordinate axes with labels and scales. (CCSS: A-CED.2)
   iii. Represent constraints by equations or inequalities, and by systems of equations and/or inequalities, and interpret solutions as viable or nonviable options in a modeling context.[21] (CCSS: A-CED.3)
   iv. Rearrange formulas to highlight a quantity of interest, using the same reasoning as in solving equations.[22] (CCSS: A-CED.4)

b. Understand solving equations as a process of reasoning and explain the reasoning. (CCSS: A-REI)
   i. Explain each step in solving a simple equation as following from the equality of numbers asserted at the previous step, starting from the assumption that the original equation has a solution. (CCSS: A-REI.1)
   ii. Solve simple rational and radical equations in one variable, and give examples showing how extraneous solutions may arise. (CCSS: A-REI.2)

c. Solve equations and inequalities in one variable. (CCSS: A-REI)
   i. Solve linear equations and inequalities in one variable, including equations with coefficients represented by letters. (CCSS: A-REI.3)
   ii. Solve quadratic equations in one variable. (CCSS: A-REI.4)
      1. Use the method of completing the square to transform any quadratic equation in x into an equation of the form $(x - p)^2 = q$ that has the same solutions. Derive the quadratic formula from this form. (CCSS: A-REI.4a)
      2. Solve quadratic equations[23] by inspection, taking square roots, completing the square, the quadratic formula and factoring, as appropriate to the initial form of the equation. (CCSS: A-REI.4b)
      3. Recognize when the quadratic formula gives complex solutions and write them as $a \pm bi$ for real numbers a and b. (CCSS: A-REI.4b)

d. Solve systems of equations. (CCSS: A-REI)
   i. Prove that, given a system of two equations in two variables, replacing one equation by the sum of that equation and a multiple of the other produces a system with the same solutions. (CCSS: A-REI.5)
   ii. Solve systems of linear equations exactly and approximately,[24] focusing on pairs of linear equations in two variables. (CCSS: A-REI.6)
   iii. Solve a simple system consisting of a linear equation and a quadratic equation in two variables algebraically and graphically.[25] (CCSS: A-REI.7)

e. Represent and solve equations and inequalities graphically. (CCSS: A-REI)
   i. Explain that the graph of an equation in two variables is the set of all its solutions plotted in the coordinate plane, often forming a curve.[26] (CCSS: A-REI.10)
   ii. Explain why the x-coordinates of the points where the graphs of the equations $y = f(x)$ and $y = g(x)$ intersect are the solutions of the equation $f(x) = g(x)$;[27] find the solutions approximately.[28]★ (CCSS: A-REI.11)
   iii. Graph the solutions to a linear inequality in two variables as a half-plane (excluding the boundary in the case of a strict inequality), and graph the solution set to a system of linear inequalities in two variables as the intersection of the corresponding half-planes. (CCSS: A-REI.12)

**Indicates a part of the standard connected to the mathematical practice of Modeling*

21st Century Skills and Readiness Competencies

Inquiry Questions:

1. What are some similarities in solving all types of equations?
2. Why do different types of equations require different types of solution processes?
3. Can computers solve algebraic problems that people cannot solve? Why?
4. How are order of operations and operational relationships important when solving multivariable equations?

Relevance and Application:

1. Linear programming allows representation of the constraints in a real-world situation identification of a feasible region and determination of the maximum or minimum value such as to optimize profit, or to minimize expense.
2. Effective use of graphing technology helps to find solutions to equations or systems of equations.

Nature of Mathematics:

1. Mathematics involves visualization.
2. Mathematicians use tools to create visual representations of problems and ideas that reveal relationships and meaning.
3. Mathematicians construct viable arguments and critique the reasoning of others. (MP)
4. Mathematicians use appropriate tools strategically. (MP)

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[1] If *f* is a function and *x* is an element of its domain, then $f(x)$ denotes the output of *f* corresponding to the input *x*. The graph of *f* is the graph of the equation $y = f(x)$. (CCSS: F-IF.1)
[2] For example, the Fibonacci sequence is defined recursively by f(0) = f(1) = 1, f(n+1) = f(n) + f(n-1) for n ≥ 1. (CCSS: F-IF.3)
[3] Key features include: intercepts; intervals where the function is increasing, decreasing, positive, or negative; relative maximums and minimums; symmetries; end behavior; and periodicity. (CCSS: F-IF.4)
[4] For example, if the function h(n) gives the number of person-hours it takes to assemble n engines in a factory, then the positive integers would be an appropriate domain for the function. (CCSS: F-IF.5)
[5] presented symbolically or as a table. (CCSS: F-IF.6)
[6] For example, identify percent rate of change in functions such as y = (1.02)t, y = (0.97)t, y = (1.01)12t, y = (1.2)t/10,. (CCSS: F-IF.8b)
[7] For example, given a graph of one quadratic function and an algebraic expression for another, say which has the larger maximum. (CCSS: F-IF.9)
[8] For example, build a function that models the temperature of a cooling body by adding a constant function to a decaying exponential, and relate these functions to the model. (CCSS: F-BF.1b)
[9] both positive and negative. (CCSS: F-BF.3)
[10] Include recognizing even and odd functions from their graphs and algebraic expressions for them. (CCSS: F-BF.3)
[11] Solve an equation of the form f(x) = c for a simple function f that has an inverse and write an expression for the inverse.
For example, $f(x) = 2x^3$ or $f(x) = (x+1)/(x-1)$ for $x \neq 1$. (CCSS: F-BF.4a)
[12] include reading these from a table. (CCSS: F-LE.2)
[13] For example, interpret $P(1+r)^n$ as the product of P and a factor not depending on P. (CCSS: A-SSE.1b)
[14] For example, see $x^4 - y^4$ as $(x^2)^2 - (y^2)^2$, thus recognizing it as a difference of squares that can be factored as $(x^2 - y^2)(x^2 + y^2)$. (CCSS: A-SSE.2)
[15] For example the expression 1.15^t can be rewritten as $(1.15^{1/12})^{12t} \approx 1.012^{12t}$ to reveal the approximate equivalent monthly interest rate if the annual rate is 15%. (CCSS: A-SSE.3c)
[16] For example, calculate mortgage payments. (CCSS: A-SSE.4)
[17] For a polynomial p(x) and a number a, the remainder on division by x – a is p(a), so p(a) = 0 if and only if (x – a) is a factor of p(x). (CCSS: A-APR.2)
[18] For example, the polynomial identity $(x^2 + y^2)^2 = (x^2 - y^2)^2 + (2xy)^2$ can be used to generate Pythagorean triples. (CCSS: A-APR.4)
[19] write $^{a(x)}/_{b(x)}$ in the form $q(x) + {}^{r(x)}/_{b(x)}$, where $a(x)$, $b(x)$, $q(x)$, and $r(x)$ are polynomials with the degree of $r(x)$ less than the degree of $b(x)$, using inspection, long division, or, for the more complicated examples, a computer algebra system. (CCSS: A-APR.6)
[20] Include equations arising from linear and quadratic functions, and simple rational and exponential functions. (CCSS: A-CED.1)
[21] For example, represent inequalities describing nutritional and cost constraints on combinations of different foods. (CCSS: A-CED.3)
[22] For example, rearrange Ohm's law V = IR to highlight resistance R. (CCSS: A-CED.4)
[23] e.g., for $x^2 = 49$. (CCSS: A-REI.4b)
[24] e.g., with graphs. (CCSS: A-REI.6)
[25] For example, find the points of intersection between the line $y = -3x$ and the circle $x^2 + y^2 = 3$. (CCSS: A-REI.7)
[26] which could be a line. (CCSS: A-REI.10)
[27] Include cases where $f(x)$ and/or $g(x)$ are linear, polynomial, rational, absolute value, exponential, and logarithmic functions. (CCSS: A-REI.11)
[28] e.g., using technology to graph the functions, make tables of values, or find successive approximations. (CCSS: A-REI.11)

3. Data Analysis, Statistics, and Probability

Data and probability sense provides students with tools to understand information and uncertainty. Students ask questions and gather and use data to answer them. Students use a variety of data analysis and statistics strategies to analyze, develop and evaluate inferences based on data. Probability provides the foundation for collecting, describing, and interpreting data.

Prepared Graduates

The prepared graduate competencies are the preschool through twelfth-grade concepts and skills that all students who complete the Colorado education system must master to ensure their success in a postsecondary and workforce setting.

Prepared Graduate Competencies in the 3. Data Analysis, Statistics, and Probability Standard are:

- Recognize and make sense of the many ways that variability, chance, and randomness appear in a variety of contexts
- Solve problems and make decisions that depend on understanding, explaining, and quantifying the variability in data
- Communicate effective logical arguments using mathematical justification and proof. Mathematical argumentation involves making and testing conjectures, drawing valid conclusions, and justifying thinking
- Use critical thinking to recognize problematic aspects of situations, create mathematical models, and present and defend solutions

Content Area: Mathematics
Standard: 3. Data Analysis, Statistics, and Probability

Prepared Graduates:
- Solve problems and make decisions that depend on understanding, explaining, and quantifying the variability in data

Grade Level Expectation: High School

Concepts and skills students master:
1. Visual displays and summary statistics condense the information in data sets into usable knowledge

Evidence Outcomes	21st Century Skills and Readiness Competencies
Students can: a. Summarize, represent, and interpret data on a single count or measurement variable. (CCSS: S-ID) i. Represent data with plots on the real number line (dot plots, histograms, and box plots). (CCSS: S-ID.1) ii. Use statistics appropriate to the shape of the data distribution to compare center (median, mean) and spread (interquartile range, standard deviation) of two or more different data sets. (CCSS: S-ID.2) iii. Interpret differences in shape, center, and spread in the context of the data sets, accounting for possible effects of extreme data points (outliers). (CCSS: S-ID.3) iv. Use the mean and standard deviation of a data set to fit it to a normal distribution and to estimate population percentages and identify data sets for which such a procedure is not appropriate. (CCSS: S-ID.4) v. Use calculators, spreadsheets, and tables to estimate areas under the normal curve. (CCSS: S-ID.4) b. Summarize, represent, and interpret data on two categorical and quantitative variables. (CCSS: S-ID) i. Summarize categorical data for two categories in two-way frequency tables. Interpret relative frequencies in the context of the data[1] (including joint, marginal, and conditional relative frequencies). Recognize possible associations and trends in the data. (CCSS: S-ID.5) ii. Represent data on two quantitative variables on a scatter plot, and describe how the variables are related. (CCSS: S-ID.6) 1. Fit a function to the data; use functions fitted to data to solve problems in the context of the data. Use given functions or choose a function suggested by the context. Emphasize linear, quadratic, and exponential models. (CCSS: S-ID.6a) 2. Informally assess the fit of a function by plotting and analyzing residuals. (CCSS: S-ID.6b) 3. Fit a linear function for a scatter plot that suggests a linear association. (CCSS: S-ID.6c) c. Interpret linear models. (CCSS: S-ID) i. Interpret the slope[2] and the intercept[3] of a linear model in the context of the data. (CCSS: S-ID.7) ii. Using technology, compute and interpret the correlation coefficient of a linear fit. (CCSS: S-ID.8) iii. Distinguish between correlation and causation. (CCSS: S-ID.9)	**Inquiry Questions:** 1. What makes data meaningful or actionable? 2. Why should attention be paid to an unexpected outcome? 3. How can summary statistics or data displays be accurate but misleading? **Relevance and Application:** 1. Facility with data organization, summary, and display allows the sharing of data efficiently and collaboratively to answer important questions such as is the climate changing, how do people think about ballot initiatives in the next election, or is there a connection between cancers in a community? **Nature of Mathematics:** 1. Mathematicians create visual and numerical representations of data to reveal relationships and meaning hidden in the raw data. 2. Mathematicians reason abstractly and quantitatively. (MP) 3. Mathematicians model with mathematics. (MP) 4. Mathematicians use appropriate tools strategically. (MP)

Content Area: Mathematics
Standard: 3. Data Analysis, Statistics, and Probability

Prepared Graduates:

- Communicate effective logical arguments using mathematical justification and proof. Mathematical argumentation involves making and testing conjectures, drawing valid conclusions, and justifying thinking

Grade Level Expectation: High School

Concepts and skills students master:

2. Statistical methods take variability into account supporting informed decisions making through quantitative studies designed to answer specific questions

Evidence Outcomes	21st Century Skills and Readiness Competencies
Students can: a. Understand and evaluate random processes underlying statistical experiments. (CCSS: S-IC) i. Describe statistics as a process for making inferences about population parameters based on a random sample from that population. (CCSS: S-IC.1) ii. Decide if a specified model is consistent with results from a given data-generating process.[4] (CCSS: S-IC.2) b. Make inferences and justify conclusions from sample surveys, experiments, and observational studies. (CCSS: S-IC) i. Identify the purposes of and differences among sample surveys, experiments, and observational studies; explain how randomization relates to each. (CCSS: S-IC.3) ii. Use data from a sample survey to estimate a population mean or proportion. (CCSS: S-IC.4) iii. Develop a margin of error through the use of simulation models for random sampling. (CCSS: S-IC.4) iv. Use data from a randomized experiment to compare two treatments; use simulations to decide if differences between parameters are significant. (CCSS: S-IC.5) v. Define and explain the meaning of significance, both statistical (using *p*-values) and practical (using effect size). vi. Evaluate reports based on data. (CCSS: S-IC.6)	**Inquiry Questions:** 1. How can the results of a statistical investigation be used to support an argument? 2. What happens to sample-to-sample variability when you increase the sample size? 3. When should sampling be used? When is sampling better than using a census? 4. Can the practical significance of a given study matter more than statistical significance? Why is it important to know the difference? 5. Why is the margin of error in a study important? 6. How is it known that the results of a study are not simply due to chance? **Relevance and Application:** 1. Inference and prediction skills enable informed decision-making based on data such as whether to stop using a product based on safety concerns, or whether a political poll is pointing to a trend. **Nature of Mathematics:** 1. Mathematics involves making conjectures, gathering data, recording results, and making multiple tests. 2. Mathematicians are skeptical of apparent trends. They use their understanding of randomness to distinguish meaningful trends from random occurrences. 3. Mathematicians construct viable arguments and critique the reasoning of others. (MP) 4. Mathematicians model with mathematics. (MP) 5. Mathematicians attend to precision. (MP)

Content Area: Mathematics
Standard: 3. Data Analysis, Statistics, and Probability

Prepared Graduates:

- Recognize and make sense of the many ways that variability, chance, and randomness appear in a variety of contexts

Grade Level Expectation: High School

Concepts and skills students master:

3. Probability models outcomes for situations in which there is inherent randomness

Evidence Outcomes

Students can:

a. Understand independence and conditional probability and use them to interpret data. (CCSS: S-CP)
   i. Describe events as subsets of a sample space[5] using characteristics (or categories) of the outcomes, or as unions, intersections, or complements of other events.[6] (CCSS: S-CP.1)
   ii. Explain that two events *A* and *B* are independent if the probability of *A* and *B* occurring together is the product of their probabilities, and use this characterization to determine if they are independent. (CCSS: S-CP.2)
   iii. Using the conditional probability of *A* given *B* as *P*(*A* and *B*)/*P*(*B*), interpret the independence of *A* and *B* as saying that the conditional probability of *A* given *B* is the same as the probability of *A*, and the conditional probability of *B* given *A* is the same as the probability of *B*. (CCSS: S-CP.3)
   iv. Construct and interpret two-way frequency tables of data when two categories are associated with each object being classified. Use the two-way table as a sample space to decide if events are independent and to approximate conditional probabilities.[7] (CCSS: S-CP.4)
   v. Recognize and explain the concepts of conditional probability and independence in everyday language and everyday situations.[8] (CCSS: S-CP.5)

b. Use the rules of probability to compute probabilities of compound events in a uniform probability model. (CCSS: S-CP)
   i. Find the conditional probability of *A* given *B* as the fraction of *B*'s outcomes that also belong to *A*, and interpret the answer in terms of the model. (CCSS: S-CP.6)
   ii. Apply the Addition Rule, P(A or B) = P(A) + P(B) – P(A and B), and interpret the answer in terms of the model. (CCSS: S-CP.7)

c. Analyze* the cost of insurance as a method to offset the risk of a situation (PFL)

**Indicates a part of the standard connected to the mathematical practice of Modeling.*

21st Century Skills and Readiness Competencies

Inquiry Questions:

1. Can probability be used to model all types of uncertain situations? For example, can the probability that the 50th president of the United States will be female be determined?
2. How and why are simulations used to determine probability when the theoretical probability is unknown?
3. How does probability relate to obtaining insurance? (PFL)

Relevance and Application:

1. Comprehension of probability allows informed decision-making, such as whether the cost of insurance is less than the expected cost of illness, when the deductible on car insurance is optimal, whether gambling pays in the long run, or whether an extended warranty justifies the cost. (PFL)
2. Probability is used in a wide variety of disciplines including physics, biology, engineering, finance, and law. For example, employment discrimination cases often present probability calculations to support a claim.

Nature of Mathematics:

1. Some work in mathematics is much like a game. Mathematicians choose an interesting set of rules and then play according to those rules to see what can happen.
2. Mathematicians explore randomness and chance through probability.
3. Mathematicians construct viable arguments and critique the reasoning of others. (MP)
4. Mathematicians model with mathematics. (MP)

[1] including joint, marginal, and conditional relative frequencies.
[2] rate of change. (CCSS: S-ID.7)
[3] constant term. (CCSS: S-ID.7)
[4] e.g., using simulation. (CCSS: S-IC.2)
For example, a model says a spinning coin falls heads up with probability 0.5. Would a result of 5 tails in a row cause you to question the model? (CCSS: S-IC.2)
[5] the set of outcomes. (CCSS: S-CP.1)
[6] "or," "and," "not". (CCSS: S-CP.1)
[7] For example, collect data from a random sample of students in your school on their favorite subject among math, science, and English. Estimate the probability that a randomly selected student from your school will favor science given that the student is in tenth grade. Do the same for other subjects and compare the results. (CCSS: S-CP.4)
[8] For example, compare the chance of having lung cancer if you are a smoker with the chance of being a smoker if you have lung cancer. (CCSS: S-CP.5)

4. Shape, Dimension, and Geometric Relationships

Geometric sense allows students to comprehend space and shape. Students analyze the characteristics and relationships of shapes and structures, engage in logical reasoning, and use tools and techniques to determine measurement. Students learn that geometry and measurement are useful in representing and solving problems in the real world as well as in mathematics.

Prepared Graduates
The prepared graduate competencies are the preschool through twelfth-grade concepts and skills that all students who complete the Colorado education system must master to ensure their success in a postsecondary and workforce setting.

Prepared Graduate Competencies in the 4. Shape, Dimension, and Geometric Relationships standard are:

- Understand quantity through estimation, precision, order of magnitude, and comparison. The reasonableness of answers relies on the ability to judge appropriateness, compare, estimate, and analyze error
- Make sound predictions and generalizations based on patterns and relationships that arise from numbers, shapes, symbols, and data
- Apply transformation to numbers, shapes, functional representations, and data
- Make claims about relationships among numbers, shapes, symbols, and data and defend those claims by relying on the properties that are the structure of mathematics
- Use critical thinking to recognize problematic aspects of situations, create mathematical models, and present and defend solutions

Content Area: Mathematics
Standard: 4. Shape, Dimension, and Geometric Relationships

Prepared Graduates:

- Apply transformation to numbers, shapes, functional representations, and data

Grade Level Expectation: High School

Concepts and skills students master:

1. Objects in the plane can be transformed, and those transformations can be described and analyzed mathematically

Evidence Outcomes	21st Century Skills and Readiness Competencies
Students can: a. Experiment with transformations in the plane. (CCSS: G-CO) i. State precise definitions of angle, circle, perpendicular line, parallel line, and line segment, based on the undefined notions of point, line, distance along a line, and distance around a circular arc. (CCSS: G-CO.1) ii. Represent transformations in the plane using[1] appropriate tools. (CCSS: G-CO.2) iii. Describe transformations as functions that take points in the plane as inputs and give other points as outputs. (CCSS: G-CO.2) iv. Compare transformations that preserve distance and angle to those that do not.[2] (CCSS: G-CO.2) v. Given a rectangle, parallelogram, trapezoid, or regular polygon, describe the rotations and reflections that carry it onto itself. (CCSS: G-CO.3) vi. Develop definitions of rotations, reflections, and translations in terms of angles, circles, perpendicular lines, parallel lines, and line segments. (CCSS: G-CO.4) vii. Given a geometric figure and a rotation, reflection, or translation, draw the transformed figure using appropriate tools.[3] (CCSS: G-CO.5) viii. Specify a sequence of transformations that will carry a given figure onto another. (CCSS: G-CO.5) b. Understand congruence in terms of rigid motions. (CCSS: G-CO) i. Use geometric descriptions of rigid motions to transform figures and to predict the effect of a given rigid motion on a given figure. (CCSS: G-CO.6) ii. Given two figures, use the definition of congruence in terms of rigid motions to decide if they are congruent. (CCSS: G-CO.6) iii. Use the definition of congruence in terms of rigid motions to show that two triangles are congruent if and only if corresponding pairs of sides and corresponding pairs of angles are congruent. (CCSS: G-CO.7) iv. Explain how the criteria for triangle congruence (ASA, SAS, and SSS) follow from the definition of congruence in terms of rigid motions. (CCSS: G-CO.8) c. Prove geometric theorems. (CCSS: G-CO) i. Prove theorems about lines and angles.[4] (CCSS: G-CO.9) ii. Prove theorems about triangles.[5] (CCSS: G-CO.10) iii. Prove theorems about parallelograms.[6] (CCSS: G-CO.11) d. Make geometric constructions. (CCSS: G-CO) i. Make formal geometric constructions[7] with a variety of tools and methods.[8] (CCSS: G-CO.12) ii. Construct an equilateral triangle, a square, and a regular hexagon inscribed in a circle. (CCSS: G-CO.13)	**Inquiry Questions:** 1. What happens to the coordinates of the vertices of shapes when different transformations are applied in the plane? 2. How would the idea of congruency be used outside of mathematics? 3. What does it mean for two things to be the same? Are there different degrees of "sameness?" 4. What makes a good definition of a shape? **Relevance and Application:** 1. Comprehension of transformations aids with innovation and creation in the areas of computer graphics and animation. **Nature of Mathematics:** 1. Geometry involves the investigation of invariants. Geometers examine how some things stay the same while other parts change to analyze situations and solve problems. 2. Mathematicians construct viable arguments and critique the reasoning of others. (MP) 3. Mathematicians attend to precision. (MP) 4. Mathematicians look for and make use of structure. (MP)

Content Area: Mathematics
Standard: 4. Shape, Dimension, and Geometric Relationships

Prepared Graduates:

- ➢ Use critical thinking to recognize problematic aspects of situations, create mathematical models, and present and defend solutions

Grade Level Expectation: High School

Concepts and skills students master:

2. Concepts of similarity are foundational to geometry and its applications

Evidence Outcomes

Students can:

a. Understand similarity in terms of similarity transformations. (CCSS: G-SRT)
   i. Verify experimentally the properties of dilations given by a center and a scale factor. (CCSS: G-SRT.1)
      1. Show that a dilation takes a line not passing through the center of the dilation to a parallel line, and leaves a line passing through the center unchanged. (CCSS: G-SRT.1a)
      2. Show that the dilation of a line segment is longer or shorter in the ratio given by the scale factor. (CCSS: G-SRT.1b)
   ii. Given two figures, use the definition of similarity in terms of similarity transformations to decide if they are similar. (CCSS: G-SRT.2)
   iii. Explain using similarity transformations the meaning of similarity for triangles as the equality of all corresponding pairs of angles and the proportionality of all corresponding pairs of sides. (CCSS: G-SRT.2)
   iv. Use the properties of similarity transformations to establish the AA criterion for two triangles to be similar. (CCSS: G-SRT.3)

b. Prove theorems involving similarity. (CCSS: G-SRT)
   i. Prove theorems about triangles.[9] (CCSS: G-SRT.4)
   ii. Prove that all circles are similar. (CCSS: G-C.1)
   iii. Use congruence and similarity criteria for triangles to solve problems and to prove relationships in geometric figures. (CCSS: G-SRT.5)

c. Define trigonometric ratios and solve problems involving right triangles. (CCSS: G-SRT)
   i. Explain that by similarity, side ratios in right triangles are properties of the angles in the triangle, leading to definitions of trigonometric ratios for acute angles. (CCSS: G-SRT.6)
   ii. Explain and use the relationship between the sine and cosine of complementary angles. (CCSS: G-SRT.7)
   iii. Use trigonometric ratios and the Pythagorean Theorem to solve right triangles in applied problems.* (CCSS: G-SRT.8)

d. Prove and apply trigonometric identities. (CCSS: F-TF)
   i. Prove the Pythagorean identity $\sin^2(\theta) + \cos^2(\theta) = 1$. (CCSS: F-TF.8)
   ii. Use the Pythagorean identity to find sin(θ), cos(θ), or tan(θ) given sin(θ), cos(θ), or tan(θ) and the quadrant of the angle. (CCSS: F-TF.8)

e. Understand and apply theorems about circles. (CCSS: G-C)
   i. Identify and describe relationships among inscribed angles, radii, and chords.[10] (CCSS: G-C.2)
   ii. Construct the inscribed and circumscribed circles of a triangle. (CCSS: G-C.3)
   iii. Prove properties of angles for a quadrilateral inscribed in a circle. (CCSS: G-C.3)

f. Find arc lengths and areas of sectors of circles. (CCSS: G-C)
   i. Derive using similarity the fact that the length of the arc intercepted by an angle is proportional to the radius, and define the radian measure of the angle as the constant of proportionality. (CCSS: G-C.5)
   ii. Derive the formula for the area of a sector. (CCSS: G-C.5)

**Indicates a part of the standard connected to the mathematical practice of Modeling*

21st Century Skills and Readiness Competencies

Inquiry Questions:

1. How can you determine the measure of something that you cannot measure physically?
2. How is a corner square made?
3. How are mathematical triangles different from triangles in the physical world? How are they the same?
4. Do perfect circles naturally occur in the physical world?

Relevance and Application:

1. Analyzing geometric models helps one understand complex physical systems. For example, modeling Earth as a sphere allows us to calculate measures such as diameter, circumference, and surface area. We can also model the solar system, galaxies, molecules, atoms, and subatomic particles.

Nature of Mathematics:

1. Geometry involves the generalization of ideas. Geometers seek to understand and describe what is true about all cases related to geometric phenomena.
2. Mathematicians construct viable arguments and critique the reasoning of others. (MP)
3. Mathematicians attend to precision. (MP)

Content Area: Mathematics
Standard: 4. Shape, Dimension, and Geometric Relationships

Prepared Graduates:

- Make claims about relationships among numbers, shapes, symbols, and data and defend those claims by relying on the properties that are the structure of mathematics

Grade Level Expectation: High School

Concepts and skills students master:

3. Objects in the plane can be described and analyzed algebraically

Evidence Outcomes	21st Century Skills and Readiness Competencies
Students can: a. Express Geometric Properties with Equations. (CCSS: G-GPE) i. Translate between the geometric description and the equation for a conic section. (CCSS: G-GPE) 1. Derive the equation of a circle of given center and radius using the Pythagorean Theorem. (CCSS: G-GPE.1) 2. Complete the square to find the center and radius of a circle given by an equation. (CCSS: G-GPE.1) 3. Derive the equation of a parabola given a focus and directrix. (CCSS: G-GPE.2) ii. Use coordinates to prove simple geometric theorems algebraically. (CCSS: G-GPE) 1. Use coordinates to prove simple geometric theorems[11] algebraically. (CCSS: G-GPE.4) 2. Prove the slope criteria for parallel and perpendicular lines and use them to solve geometric problems.[12] (CCSS: G-GPE.5) 3. Find the point on a directed line segment between two given points that partitions the segment in a given ratio. (CCSS: G-GPE.6) 4. Use coordinates and the distance formula to compute perimeters of polygons and areas of triangles and rectangles.★ (CCSS: G-GPE.7) **Indicates a part of the standard connected to the mathematical practice of Modeling*	**Inquiry Questions:** 1. What does it mean for two lines to be parallel? 2. What happens to the coordinates of the vertices of shapes when different transformations are applied in the plane? **Relevance and Application:** 1. Knowledge of right triangle trigonometry allows modeling and application of angle and distance relationships such as surveying land boundaries, shadow problems, angles in a truss, and the design of structures. **Nature of Mathematics:** 1. Geometry involves the investigation of invariants. Geometers examine how some things stay the same while other parts change to analyze situations and solve problems. 2. Mathematicians make sense of problems and persevere in solving them. (MP) 3. Mathematicians construct viable arguments and critique the reasoning of others. (MP)

Content Area: Mathematics
Standard: 4. Shape, Dimension, and Geometric Relationships

Prepared Graduates:

- Make claims about relationships among numbers, shapes, symbols, and data and defend those claims by relying on the properties that are the structure of mathematics

Grade Level Expectation: High School

Concepts and skills students master:

4. Attributes of two- and three-dimensional objects are measurable and can be quantified

Evidence Outcomes	21st Century Skills and Readiness Competencies
Students can: a. Explain volume formulas and use them to solve problems. (CCSS: G-GMD) i. Give an informal argument[13] for the formulas for the circumference of a circle, area of a circle, volume of a cylinder, pyramid, and cone. (CCSS: G-GMD.1) ii. Use volume formulas for cylinders, pyramids, cones, and spheres to solve problems.★ (CCSS: G-GMD.3) b. Visualize relationships between two-dimensional and three-dimensional objects. (CCSS: G-GMD) i. Identify the shapes of two-dimensional cross-sections of three-dimensional objects, and identify three-dimensional objects generated by rotations of two-dimensional objects. (CCSS: G-GMD.4) **Indicates a part of the standard connected to the mathematical practice of Modeling*	**Inquiry Questions:** 1. How might surface area and volume be used to explain biological differences in animals? 2. How is the area of an irregular shape measured? 3. How can surface area be minimized while maximizing volume? **Relevance and Application:** 1. Understanding areas and volume enables design and building. For example, a container that maximizes volume and minimizes surface area will reduce costs and increase efficiency. Understanding area helps to decorate a room, or create a blueprint for a new building. **Nature of Mathematics:** 1. Mathematicians use geometry to model the physical world. Studying properties and relationships of geometric objects provides insights in to the physical world that would otherwise be hidden. 2. Mathematicians make sense of problems and persevere in solving them. (MP) 3. Mathematicians construct viable arguments and critique the reasoning of others. (MP) 4. Mathematicians model with mathematics. (MP)

Content Area: Mathematics
Standard: 4. Shape, Dimension, and Geometric Relationships

Prepared Graduates:

- Use critical thinking to recognize problematic aspects of situations, create mathematical models, and present and defend solutions

Grade Level Expectation: High School

Concepts and skills students master:

5. Objects in the real world can be modeled using geometric concepts

Evidence Outcomes	21st Century Skills and Readiness Competencies
Students can: a. Apply geometric concepts in modeling situations. (CCSS: G-MG) i. Use geometric shapes, their measures, and their properties to describe objects.[14]★ (CCSS: G-MG.1) ii. Apply concepts of density based on area and volume in modeling situations.[15]★ (CCSS: G-MG.2) iii. Apply geometric methods to solve design problems.[16]★ (CCSS: G-MG.3) **Indicates a part of the standard connected to the mathematical practice of Modeling*	**Inquiry Questions:** 1. How are mathematical objects different from the physical objects they model? 2. What makes a good geometric model of a physical object or situation? 3. How are mathematical triangles different from built triangles in the physical world? How are they the same? **Relevance and Application:** 1. Geometry is used to create simplified models of complex physical systems. Analyzing the model helps to understand the system and is used for such applications as creating a floor plan for a house, or creating a schematic diagram for an electrical system. **Nature of Mathematics:** 1. Mathematicians use geometry to model the physical world. Studying properties and relationships of geometric objects provides insights in to the physical world that would otherwise be hidden. 2. Mathematicians make sense of problems and persevere in solving them. (MP) 3. Mathematicians reason abstractly and quantitatively. (MP) 4. Mathematicians look for and make use of structure. (MP)

[1] e.g., transparencies and geometry software. (CCSS: G-CO.2)
[2] e.g., translation versus horizontal stretch. (CCSS: G-CO.2)
[3] e.g., graph paper, tracing paper, or geometry software. (CCSS: G-CO.5)
[4] Theorems include: vertical angles are congruent; when a transversal crosses parallel lines, alternate interior angles are congruent and corresponding angles are congruent; points on a perpendicular bisector of a line segment are exactly those equidistant from the segment's endpoints. (CCSS: G-CO.9)
[5] Theorems include: measures of interior angles of a triangle sum to 180°; base angles of isosceles triangles are congruent; the segment joining midpoints of two sides of a triangle is parallel to the third side and half the length; the medians of a triangle meet at a point. (CCSS: G-CO.10)
[6] Theorems include: opposite sides are congruent, opposite angles are congruent, the diagonals of a parallelogram bisect each other, and conversely, rectangles are parallelograms with congruent diagonals. (CCSS: G-CO.11)
[7] Copying a segment; copying an angle; bisecting a segment; bisecting an angle; constructing perpendicular lines, including the perpendicular bisector of a line segment; and constructing a line parallel to a given line through a point not on the line. (CCSS: G-CO.12)
[8] compass and straightedge, string, reflective devices, paper folding, dynamic geometric software, etc. (CCSS: G-CO.12)
[9] Theorems include: a line parallel to one side of a triangle divides the other two proportionally, and conversely; the Pythagorean Theorem proved using triangle similarity. (CCSS: G-SRT.4)
[10] Include the relationship between central, inscribed, and circumscribed angles; inscribed angles on a diameter are right angles; the radius of a circle is perpendicular to the tangent where the radius intersects the circle. (CCSS: G-C.2)
[11] For example, prove or disprove that a figure defined by four given points in the coordinate plane is a rectangle; prove or disprove that the point $(1, \sqrt{3})$ lies on the circle centered at the origin and containing the point $(0, 2)$. (CCSS: G-GPE.4)
[12] e.g., find the equation of a line parallel or perpendicular to a given line that passes through a given point. (CCSS: G-GPE.5)
[13] Use dissection arguments, Cavalieri's principle, and informal limit arguments. (CCSS: G-GMD.1)
[14] e.g., modeling a tree trunk or a human torso as a cylinder. (CCSS: G-MG.1)
[15] e.g., persons per square mile, BTUs per cubic foot. (CCSS: G-MG.2)
[16] e.g., designing an object or structure to satisfy physical constraints or minimize cost; working with typographic grid systems based on ratios. (CCSS: G-MG.3)

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