Algebra II Assessments

performance tasks aligned to the Texas standards

The Charles A. Dana Center
at The University of Texas at Austin

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About The University of Texas at Austin Charles A. Dana Center

The Charles A. Dana Center supports teachers, education leaders, and policymakers in strengthening education. As a research unit of The University of Texas at Austin's College of Natural Sciences, the Dana Center maintains a special emphasis on mathematics and science education. The Dana Center's mission is to strengthen the mathematics and science preparation and achievement of all students through supporting alignment of all the key components of mathematics and science education, prekindergarten–16: the state standards, accountability system, assessment, and teacher preparation. We focus our efforts on providing resources to help local communities meet the demands of the education system—by working with leaders, teachers, and students through our Instructional Support System; by strengthening mathematics and science professional development; and by publishing and disseminating mathematics and science education resources.

About the development of this book

The Charles A. Dana Center has developed this standards-aligned mathematics education resource for mathematics teachers.

The development and production of the first edition of *Algebra II Assessments* was supported in part by the Texas Education Agency, the National Science Foundation under cooperative agreement #ESR-9712001, and the Charles A. Dana Center at The University of Texas at Austin. The development and production of this second edition was supported by the Charles A. Dana Center. Any opinions, findings, conclusions, or recommendations expressed in this material are those of the author(s) and do not necessarily reflect the views of the Texas Education Agency, the National Science Foundation, or The University of Texas at Austin.

The second edition updates the Texas Essential Knowledge and Skills statements and alignment charts to align with the state's 2005–06 revisions to the Mathematics TEKS. The assessments themselves have not been updated. The second edition has also been re-edited for clarity and to correct some minor errors.

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TEKS and TAKS Resources

The mathematics Texas Essential Knowledge and Skills (TEKS) were developed by the state of Texas to clarify what all students should know and be able to do in mathematics in kindergarten through grade 12. Districts are required to provide instruction that is aligned with the mathematics TEKS, which were originally adopted by the State Board of Education in 1997 and implemented statewide in 1998. Revisions to the Mathematics TEKS were adopted in 2005–06 and implemented starting in Fall 2006. The mathematics TEKS also form the objectives and student expectations for the mathematics portion of the Texas Assessment of Knowledge and Skills (TAKS).

The mathematics TEKS can be downloaded in printable format, free of charge, from the Dana Center's Mathematics TEKS Toolkit website **(www.mathtekstoolkit.org).** Perfect-bound and spiral-bound versions of the mathematics and science TEKS booklets are available for a fee (to cover the costs of production) from the Charles A. Dana Center at The University of Texas at Austin **(www.utdanacenter.org).**

Resources for implementing the mathematics TEKS are available through the Charles A. Dana Center, regional education service centers, and the Texas Education Agency. Online resources can be found at in the Dana Center's Mathematics TEKS Toolkit at **www.mathtekstoolkit.org.**

The following products and services are also available from the Dana Center at **www.utdanacenter.org/catalog:**

- Revised Mathematics TEKS booklets and Mathematics TEKS charts for K–8 and 6–12
- *Mathematics Standards in the Classroom: Resources for Grades 3–5* and *6–8*
- *Middle School Mathematics Assessments: Proportional Reasoning*
- *Geometry Assessments*
- *Algebra I Assessments*
- Professional development for mathematics and science teachers
- Professional development modules for educators, including administrators, teachers, and other leaders

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Introduction

The importance of mathematics assessment

The Dana Center developed *Algebra II Assessments* as a resource for teachers to use to provide ongoing assessment integrated with Algebra II mathematics instruction.

The National Council of Teachers of Mathematics (2000) lists as one of its six principles for school mathematics that "Assessment should support the learning of important mathematics and furnish useful information to both teachers and students."[1]

Further, NCTM (1995)[2] identified the following six standards to guide classroom assessment:

The Mathematics Standard:	Assessment should reflect the mathematics that all students need to know and be able to do.
The Learning Standard:	Assessment should enhance mathematics learning.
The Equity Standard:	Assessment should promote equity.
The Openness Standard:	Assessment should be an open process.
The Inferences Standard:	Assessment should promote valid inferences about mathematics learning.
The Coherence Standard:	Assessment should be a coherent process.

What are the *Algebra II Assessments*?

Teachers can use these *Algebra II Assessments* to provide ongoing assessment integrated with Algebra II instruction. The performance tasks, which embody what all students need to know and be able to do in a high school Algebra II course, may be used for formative, summative, or ongoing assessment. The tasks are designed to diagnose students' understanding of concepts and their procedural knowledge, rather than simply determine whether they reached the "right" or "wrong" answers. Teachers should assess frequently to monitor individual performance and guide instruction.

The purpose of these assessments is to

- clarify for teachers, students, and parents what is being taught and learned in Algebra II,
- help teachers gain evidence of student insight and misconceptions to use as a basis for instructional decisions, and
- provide teachers with questioning strategies to guide instruction and enhance student learning.

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[1] National Council of Teachers of Mathematics (2000). *Principles and Standards for School Mathematics*. Reston, VA: National Council of Teachers of Mathematics. (Summary available at standards.nctm.org.)

[2] National Council of Teachers of Mathematics. (1995). *Assessment Standards for School Mathematics.* Reston, VA: National Council of Teachers of Mathematics, pages 11, 13, 15, 17, 19, and 21.

What's new?

The first edition of Algebra II Assessments was published in 2003. In this second edition, we have included the 2005 revised secondary mathematics Texas Essential Knowledge and Skills and updated the TEKS alignment charts.

How do the assessments support TEKS-based instruction?

Each performance task in the assessments book

- is aligned with the revised Algebra II TEKS student expectations,
- is aligned with the grade 11 exit-level Texas Assessment of Knowledge and Skills (TAKS) objectives, and
- is aligned with a Dana Center TEXTEAMS institute—Practice-Based Professional Development: *Algebra II Assessments*.

How are the assessments structured?

Teachers may use these assessments formatively or summatively, for individual students or groups of students. Each assessment

- includes a performance task,
- is aligned with the Algebra II mathematics TEKS knowledge and skills as well as student expectations,
- is aligned with the TAKS objectives,
- includes "scaffolding" questions that the teacher may use to help the student analyze the problem,
- provides a sample solution,* and
- includes extension questions to bring out additional mathematical concepts in a summative discussion of solutions to the problem.

*The sample solution is only one way that a performance task may be approached and is not necessarily the "best" solution. For many of the tasks, there are other approaches that will also provide a correct analysis of the task. The authors have attempted to illustrate a variety of mathematical approaches in the various sample solutions. Several of the assessments also include samples of anonymous student work.

Algebra II Assessments presents performance tasks based on the five strands in the Algebra II TEKS—we have subdivided the tasks into eight categories:

1. Foundations of Functions
2. Transformations
3. Linear Systems
4. Quadratic Functions
5. Square Root Functions
6. Exponential and Logarithmic Functions
7. Rational Functions
8. Conics

What is the solution guide?

The solution guide is a one-page problem-solving checklist in the front of the book that teachers may use to track what is necessary for a student to give a complete solution for a performance task—because students need to know what criteria they are expected to meet in their solutions. When assigning the performance task, the teacher can give students copies of a solution guide customized with marks indicating which of the criteria should be considered in the performance task analysis. For most performance tasks, all the criteria will be important, but initially the teacher may want to focus on only two or three criteria.

On the page before a student work sample, we provide comments on some of the solution criteria that are evident from the student's solution. The professional development experience described below will help the teacher use the solution guide in the classroom and will also help guide the teacher to use other assessment evaluation tools.

Where can I get a copy of *Algebra II Assessments*, second edition?

Algebra II Assessments, second edition, is available for purchase in book or CD-ROM format by visiting our online catalog at **www.utdanacenter.org/catalog**. The second edition of *Algebra II Assessments*, or a portion thereof, is also available free of charge from the Dana Center's mathematics toolkit at **www.mathtekstoolkit.org**.

Is professional development available to support the *Algebra II Assessments*?

Yes. The Dana Center has developed a three-day TEXTEAMS institute that focuses on the implementation of the assessments: *TEXTEAMS Practice-Based Professional Development: Algebra II Assessments.*

Participants in these institutes will

- experience selected assessments,
- analyze student work to evaluate student understanding,
- consider methods for evaluating student work,
- view a video of students working on the assessments,
- develop strategies for classroom implementation, and
- consider how the assessments support the TAKS.

Contact your local school district or regional service center to find out when these institutes are offered.

Algebra II Solution Guide

The teacher will mark the criteria to be considered in the solution of this particular problem	Criteria	Check if solution satisfies this criteria
	Describes functional relationships	
	Defines variables appropriately using correct units	
	Interprets functional relationships correctly	
	Uses multiple representations (such as tables, graphs, symbols, verbal descriptions, concrete models) and makes connections among them	
	Demonstrates algebra concepts, processes, and skills	
	Interprets the reasonableness of answers in the context of the problem	
	Communicates a clear, detailed, and organized solution strategy	
	States a clear and accurate solution using correct units	
	Uses correct terminology and notation	
	Uses appropriate tools	

Assessment Alignment to TEKS and TAKS

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Data Dilemma	27	2A.4A, 2A.4B		1, 2, 5, 9, 10
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Assessment Title	Page #	TEKS Focus	Other TEKS	Connections to TAKS
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TEKS and TAKS Alignment to Assessment

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2A.10D, 2A.10F	2A.1B, 2A.4A, 2A.4B	1, 2, 5, 9, 10	Saline Solution	295
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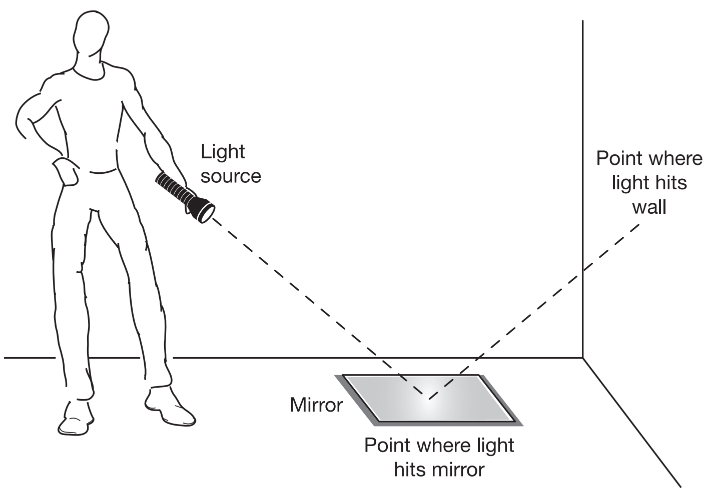
2A.5B, 2A.5C	2A.8B	2, 5, 10	Kalotonic Kaper	345

Chapter 1:
Foundations of Functions

Hit the Wall

You are playing a game in which you shine a light into a mirror on the floor while trying to aim the beam of reflected light at a target on the wall. Being a student of mathematics, and especially modeling, you realize there may be a mathematical relationship between the distance of the mirror from the wall and the height at which the light beam hits the wall. Perform the following activity to test this belief.

You will need a narrow-beamed flashlight or a laser pointer, a small mirror, and at least three group members, including yourself. The three jobs for the group members are: a shiner, a marker, and an observer. Place the mirror on the floor 0.5 feet from the wall. Have the shiner move 2 feet away from the mirror in the opposite direction from the wall and stand there. The shiner will hold the light at waist height and shine it onto the mirror so that the beam of light reflects on the wall.



The marker will measure the height at which the light hits on the wall. Record the height at the middle of the beam of reflected light on the wall. Record these values on the table.

Move *everything* 0.5 feet further from the wall and repeat the measurements. The observer's job is to make sure the shiner holds the light source in the same manner each time.

Distance between mirror and wall	Height of reflected light from floor
0.5 ft	
1 ft	
1.5 ft	
2 ft	
2.5 ft	

1. Describe the dependent and independent variables in this situation.
2. What is a reasonable domain and range for a model for this situation?
3. Create a scatterplot of your data. Does there appear to be a relationship between the distance and the height?
4. What parent function has the same appearance as the data you plotted?
5. Using the parent function you selected, develop a model for the relationship between distance from the wall and height on the wall. Graph your model on the same graph as your original plot of the data to determine its reasonableness.
6. Use your model to calculate how high up the wall the reflected light would hit if the mirror were 4 feet from the wall. What if the mirror were 6 feet from the wall?
7. Test your model by using 3 feet and 5 feet as vertical heights.
8. How could you use this technique to measure the height of any object, such as the top of a flagpole, the height of the ceiling, or the crossbar of the goal on the football field?

Notes

Materials:

Graphing calculator
Data sheet or meter stick
Laser pointer or narrow beam flash light
Mirror

Algebra II TEKS Focus:

(2A.1) Foundations for functions. The student uses properties and attributes of functions and applies functions to problem situations.

The student is expected to:

(A) identify the mathematical domains and ranges of functions and determine reasonable domain and range values for continuous and discrete situations.

(B) collect and organize data, make and interpret scatterplots, fit the graph of a function to the data, interpret the results, and proceed to model, predict, and make decisions and critical judgments.

Additional Algebra II TEKS:

(2A.2) Foundations for functions. The student understands the importance of the skills required to manipulate symbols in order to solve problems and uses the necessary algebraic skills required to simplify algebraic expressions and solve

Scaffolding Questions:

- If you shine a light on a mirror at an angle, what happens to the reflected light?
- What relationship is there between the angle at which the light hits the mirror and the angle that is made by the reflected light?
- In this activity are there any restrictions on where on the floor the mirror can be placed?
- As you move the mirror away from the wall, what happens to the reflected light?
- Why are you measuring to the middle of the reflected light?
- What would be different about using a laser pointer and not a flashlight?
- Once the light reaches the top of the wall, what happens if you move the mirror back farther?
- If you don't want the light to hit the ceiling, is there a limit to how far back the mirror can be moved?

Sample Solutions:

For sample solutions the following data will be used. This table can be distributed to students if the data collection part of the activity will not occur during the class period.

Distance between mirror and wall	Height of reflected light from floor
0.5 ft	0.75 ft
1 ft	1.5 ft
1.5 ft	2.25 ft
2 ft	3 ft
2.5 ft	3.75 ft

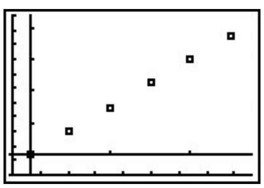
1. The height of the reflected light from the floor depends upon the distance from the wall.

2. The domain and range depends on the size of the room. The range can be no larger than the ceiling height. The domain is also dependent on ceiling height. For a classroom with a 10-foot ceiling, the domain could be restricted to be no larger than 7 feet, because after that the light source would not be reflected on the wall. If the students do not take this into account, room size would be a very reasonable value.

   Domain $0 \le x \le 20$

   Range $0 \le y \le 10$

3. The scatterplot is shown below. There does appear to be a relationship between the two sets of data, because the points seem to lie on a straight line.



4. The parent function for the data would be the linear parent function $y = x$.

5. Use two points in the data to determine the slope. The two points used below are (1.5, 2.25) and (0.5, 0.75).

$$m = \frac{2.25 - 0.75}{1.5 - 0.5} = 1.5$$

   Use the slope and one of the points to determine the value of the *y*-intercept.

$$\begin{aligned} y &= 1.5x + b \\ 2.25 &= 1.5(1.5) + b \\ 0 &= b \end{aligned}$$

equations and inequalities in problem situations.

The student is expected to:

(A) use tools including factoring and properties of exponents to simplify expressions and to transform and solve equations.

Connection to TAKS:

Objective 1: The student will describe functional relationships in a variety of ways.

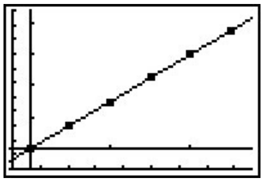
Objective 2: The student will demonstrate an understanding of the properties and attributes of functions.

Objective 3: The student will demonstrate an understanding of linear functions.

Objective 4: The student will formulate and use linear equations and inequalities.

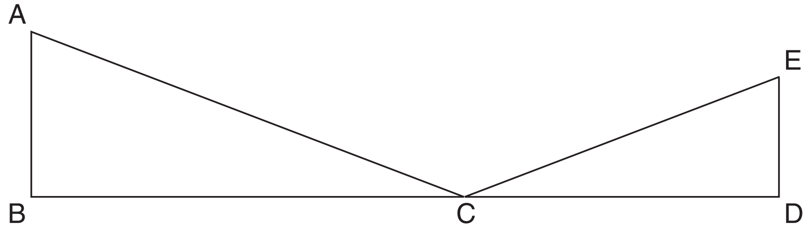
Objective 10: The student will demonstrate an understanding of the mathematical processes and tools used in problem solving.

The rule for the line is $y = 1.5x$. The rule is graphed with the data below.



Note that an alternate approach is to use a geometric model.

Draw a diagram to illustrate the problem and use geometry to explain why the relationship between the distance of the mirror and the height of the light on the wall is linear. AB is the height of the light on the wall. BC is the distance between the wall and the mirror.



The angle of incidence is equal to the angle or reflection.
Thus, $\angle ACB \cong \angle ECD$. $\angle ABC$ and $\angle EDC$ are both right angles and, thus, are congruent. Therefore $\Delta ABC \sim \Delta EDC$.

Since these two triangles are similar, $\frac{AB}{BC} = \frac{ED}{DC}$. Using values from the collected data, when AB = 1.5, BC = 1, DC = x and ED = y. Thus, $\frac{AB}{BC} = \frac{ED}{DC}$ becomes $\frac{1.5}{1} = \frac{y}{x}$ or $y = 1.5x$.

6. The given distances are represented by x in the equation $y = 1.5x$.

$$y = 1.5(4) = 6 \text{ ft}$$
$$y = 1.5(6) = 9 \text{ ft}$$

7. In order for the reflection to hit 3 feet up the wall, the mirror would have to be 2 feet away. This was one of the collected points.

   To hit 5 feet up the wall,

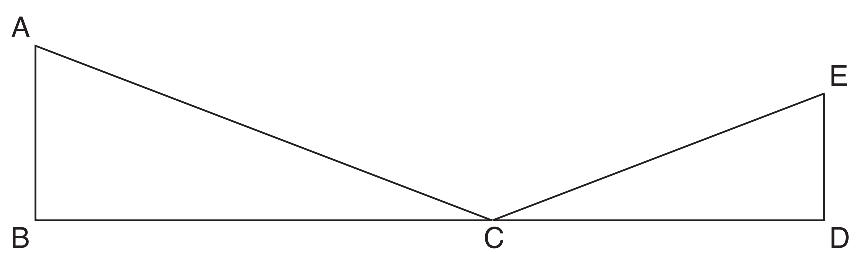
$$y = 1.5x$$
$$5 = 1.5x$$
$$\frac{5}{1.5} = x$$
$$x = 3\frac{1}{3}\text{ ft}$$

When you actually perform this "test" of the model you realize it is a reasonable model.

8. Since you probably could not see a light beam if you were outside, you would need to use your eyesight in the same manner that you used the light. Stand 2 feet away from a mirror that is placed on the ground by the object you want to measure. Move both the mirror and yourself away from the object in question until the top of the object is seen in the mirror. The height of the object, *y*, can be found by the formula, $y = \left(\frac{k}{2}\right)x$ in feet, where *k* = the distance of your eye above the ground and *x* = distance from the mirror to the base of the object you are trying to measure.

Extension Questions:

- For the situation in problem 5, at what angle was the light hitting the mirror?



From right triangle trigonometry and the collected data,

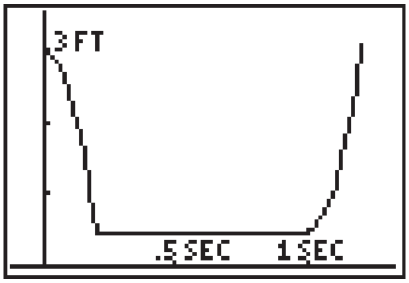
$$\tan(\angle ACB) = \frac{\text{AB}}{\text{BC}}$$
$$\tan(\angle ACB) = 1.5$$
$$m\angle ACB \approx 56.3^\circ$$

- Suppose that instead of both you and the mirror being moved to make each new measurement, the mirror stays in place and only you move. Draw a diagram and determine what family of functions could be used as a model for the relationship in this case between the height of the reflection on the wall, *y,* and your distance from the mirror, *x*.

Using the diagram from the first extension problem, AB and DC would both be constants (k_1 and k_2 respectively). BC = x and ED = y. The relationship now becomes $\frac{AB}{BC} = \frac{ED}{DC}$ *which gives* $\frac{k_1}{x} = \frac{y}{k_2}$ *or* $xy = k$. *The relationship would be an inverse relationship.*

Walk the Yo-Yo

You are playing with a yo-yo during play practice. One complete "trip" of the yo-yo is shown below. The horizontal axis represents time and the vertical axis represents distance from the floor.



1. The plot of one complete cycle of the yo-yo appears to have three parts. Describe what is happening during each phase.
2. The plot shows "time" as the independent variable. Is this reasonable? Explain your answer.
3. What are a reasonable domain and range of one cycle of the yo-yo as illustrated in the graph?

The director calls for you to come up to the stage. You walk across the room and climb four 9-inch steps up to the top of the stage. You continue to play with your yo-yo. One minute after you started playing with your yo-yo, you are on the stage and put the toy back in your pocket.

4. What is the domain and range of the total action of moving from your original position to the stage until you put the yo-yo in your pocket? Explain your answer.
5. If one complete cycle of the yo-yo takes 1.2 seconds, how many times did the yo-yo reach the bottom of its cycle during your walk to the stage?

Notes

Materials:

Yo-yo (optional; teachers may want to demonstrate what a yo-yo is and how it works).

Algebra II TEKS Focus:
(2A.1) Foundations for functions. The student uses properties and attributes of functions and applies functions to problem situations.

The student is expected to:

(A) identify the mathematical domains and ranges of functions and determine reasonable domain and range values for continuous and discrete situations.

Additional Algebra II TEKS:
None

Scaffolding Questions:

- Exactly what is a yo-yo?
- Do you have to do anything to get the yo-yo to come back up to your hand?
- Do you just drop the yo-yo, or do you "toss" it to get it to go down?
- Can it stay at the bottom of the string?
- What is the maximum height of the yo-yo according to the given graph?
- Describe how to use the graph to determine the time for one cycle.
- How is the maximum height affected by your moving up the four steps?
- What happens with the yo-yo that produces the horizontal line in the graph?

Sample Solutions:

1. For each cycle, the yo-yo goes down from the student's hand and stays at the bottom of the string for approximately a second; then it returns to the student's hand.
2. Yes. The yo-yo is moving as a result of time advancing. The motion of the yo-yo does not make time move on.
3. As illustrated, the domain is approximately $0 \leq x \leq 1.3$ (in seconds). The range is approximately $0.5 \leq y \leq 3$ (in feet).
4. The total number of seconds for the walk is 60 seconds.

   Domain $0 \leq x \leq 60$ (in seconds)

   The lowest height from the graph is about 0.5 feet. The maximum height is the height of 3 feet plus the four 9-inch steps.

Range $0.5 \leq y \leq 3 + 4 \bullet \frac{9}{12}$ (in feet)

$0.5 \leq y \leq 6$

5. The yo-yo reaches the bottom of the string once in every cycle. To determine the number of cycles in 60 seconds, divide 60 by 1.2.

$$\frac{60}{1.2} = 50 \text{ times}$$

Extension Questions:

- Using the plot of one cycle that was provided, does the yo-yo speed up or slow down as it goes down? Explain how you can visually tell.

  It speeds up. The slope is getting steeper as it falls.

- Using the plot of one cycle that was provided, does the yo-yo speed up or slow down as it returns to your hand? Explain how you can visually tell.

  It is gaining speed. The illustrated slope is again becoming steeper as it returns.

- From what you know about slopes, projectiles, and falling objects, do you think it is reasonable that the plot for one cycle of the yo-yo appears to be gaining speed as it returns to your hand?

  No, objects should be slowing as they rise because of gravity.

- Why do you think you were not asked to sketch a graph of the relationship between the height of the yo-yo and the time that has elapsed over 1 minute?

  As shown in the answer to problem 5, the yo-yo would have gone up and down 50 times. You also do not know how long it took to reach the steps.

Connection to TAKS:

Objective 1: The student will describe functional relationships in a variety of ways.

Objective 2: The student will demonstrate an understanding of the properties and attributes of functions.

Pizza Wars

1. In a recent advertisement, the pizza restaurant Little Nero's claimed that their new giant pizza is 65% bigger than the large pizza at their main competitor, Donatello's.

    a. If "65% bigger" means that the radius is 65% bigger, how much bigger than the total area of a Donatello's large pizza is the total area of a Little Nero's giant pizza? (Assume both pizzas are circular in shape.)

    b. If "65% bigger" means that the area is 65% bigger, how much bigger than the radius of a Donatello's pizza is the radius of a Little Nero's giant pizza? (Assume both pizzas are circular in shape.)

2. Which of the interpretations used above for "65% bigger" is the more likely interpretation in your opinion? Briefly explain.

3. The diameter of a large pizza at Donatello's is 14 inches. The diameter of a giant pizza at Little Nero's is 18 inches. Based on this additional information, which of the interpretations above for "65% bigger" is the correct interpretation?

Notes

Materials:

None required.

Algebra II TEKS Focus:
(2A.2) Foundations for functions. The student understands the importance of the skills required to manipulate symbols in order to solve problems and uses the necessary algebraic skills required to simplify algebraic expressions and solve equations and inequalities in problem situations.

The student is expected to:

(A) use tools including factoring and properties of exponents to simplify expressions and to transform and solve equations.

Additional Algebra II TEKS:
None

Scaffolding Questions:

- When one quantity is 65% larger than another quantity, what is the ratio of the larger quantity to the smaller quantity?
- How will this ratio help you express the larger quantity in terms of the smaller quantity?
- Can you express the area of the giant pizza in terms of the area of the large pizza?
- Do you need to solve for the radius in terms of area?
- Would it help to experiment with constants rather than variables?

Sample Solutions:

1. a. Let r represent the radius of a large pizza at Donatello's. Then $1.65r$ is the radius of a giant pizza at Little Nero's. The area of a large pizza at Donatello's is πr^2. The area of a giant pizza at Little Nero's, in terms of r, is given by:

   $\pi(1.65r)^2 = \pi(1.65)^2(r)^2 = 2.7225(\pi r^2)$ or 2.7225 (times the area of a large pizza at Donatello's)

   This represents a 172.25% increase in area. In other words, the area of a giant pizza at Little Nero's is 172.25% larger than the area of a large pizza at Donatello's.

   b. Let A_D represent the area of a large pizza at Donatello's. Then A_N is the area of the Little Nero's giant pizza. The area of a giant pizza at Little Nero's is $1.65\ A_D$.

   Let the radius at Donatello's be r_D.

   $$A_D = \pi r_D^2$$
   $$A_N = 1.65 A_D = 1.65\pi r_D^2$$

   The area of the Little Nero's giant pizza can be expressed in terms of the radius, r_N. The relationship between the two radii is determined as follows:

$$A_N = \pi r_N^2$$
$$1.65\pi r_D^2 = \pi r_N^2$$
$$r_N^2 = 1.65 r_D^2$$
$$r_N = \sqrt{1.65 r_D^2} = \sqrt{1.65}\, r_D$$
$$r_N \approx 1.285 r_D$$

The radius of the Little Nero's giant pizza is 1.2845 times the radius of a large pizza at Donatello's.

This represents a 28.45% increase in radius. In other words, the radius of a giant pizza at Little Nero's is 28.45% larger than the radius of a large pizza at Donatello's.

2. Answers will vary. Some may say that a 172% increase seems too big for the difference between a large pizza at Donatello's and a giant pizza at Little Nero's. Others may say that there should be such a big difference between a "giant" pizza and a "large" pizza.

3. If $d = 14$, then $A = \pi(7)^2 = 49\pi \approx 154$ square inches.

   If $d = 18$, then $A = \pi(9)^2 = 81\pi \approx 254.5$ square inches.

   Since $\frac{9}{7} \approx 1.2857$, the radius of the giant pizza at Little Nero's is only about 29% larger than the radius of the large pizza at Donatello's.

   Since $\frac{254.5}{154} \approx 1.6526$, the area of the giant pizza at Little Nero's is about 65% larger than the area of the large pizza at Donatello's.

   Based on the additional information, it is clear that "65% bigger" means 65% bigger in area.

Extension Questions:

- What if the pizzas are square pan pizzas? If "65% bigger" means a 65% longer side length, how much larger is the area of the giant pizza compared to the area of the large pizza? If "65% bigger" means a 65% larger area, how much longer is a side of the giant pan compared to a side of the large pan?

Connection to TAKS:

Objective 1: The student will describe functional relationships in a variety of ways.

Objective 2: The student will demonstrate an understanding of the properties and attributes of functions.

Objective 5: The student will demonstrate an understanding of quadratic and other nonlinear functions.

Objective 9: The student will demonstrate an understanding of percents, proportional relationships, probability, and statistics in application problems.

Objective 10: The student will demonstrate an understanding of the mathematical processes and tools used in problem solving.

If $A = s^2$ gives the area of the large pizza, then $A = (1.65s)^2$ gives the area of the giant pizza. Since $(1.65s)^2 = 2.7225s^2$, the results are the same as with circular pizzas. The area of the giant pizza is about 172% larger than the area of the large pizza. Likewise, since $s = \sqrt{A}$, a 65% increase in area results in a side length of $\sqrt{1.65A} \approx 1.2845\sqrt{A}$, and the results are again the same as with circular pizzas. The side length of the giant pizza is about 28% longer than the side length of the large pizza.

Pizza Wars

1.

a.) original radius = x

Donatello - original area = $3.1416x^2$

Little Nero - changed radius = $1.65x$

changed area = $8.553x^2$

$\frac{8.553x^2}{3.1416x^2}$ = 2.7225 times bigger

b.) original area = $3.1416x^2$

original radius = x

Changed area = $1.65 \cdot 3.1416x^2$

$= 5.184x^2$

Changed radius = $\sqrt{\frac{5.184x^2}{\pi}}$

$= \sqrt{1.65x^2}$

$= 1.285x$

1.285 times larger

2.) Most likely the "65% bigger" phrase applies to the area being 65% bigger (1.65 x) because if they were speaking about the radius being 65% bigger then the area would have been 2.7225 times area of Donatello's pizza. A pizza this large would cost Little Nero's more money to make the pizza and so they would have a significantly larger price. Therefore, Little Nero's would have made the area 65% bigger so they could still price the pizza so that it could compete with Donatello's large pizza.

3.) The area is 65% bigger

Donatello's... $D = 14in$

$R = 7in$

$A = \pi r^2$

$= \pi \cdot 7^2$

$= 153.94$

65% bigger for area means... 1.65×153.94

$= 253.998$

Little Nero's... $D = 18$

$R = 9$

$A = \pi r^2$

$= \pi \cdot 9^2$

$= 254$

✓ the same

∴ 65% bigger area

Catch It!

Alexa and Juan are tossing a basketball underhand back and forth in the gym. They are standing 10 feet apart. It takes 2 seconds for the ball to go from one participant to the other. They play like this for 5 minutes.

1. Is there a functional relationship between the height of the ball from the ground and time for a single toss?
2. Which variable would be the independent variable? Dependent?
3. Sketch a possible graph of one toss.
4. What would be a reasonable domain and range for this single toss? Explain how you determined the possible values for the range.
5. What would it mean if (0, 0) were a point on the graph that someone drew for this situation?
6. Suppose 0 was in the range. What would that mean?
7. Suppose that $y = 0$ represents ground level and $x = 0$ when the ball is first thrown. Does this match the graph that you drew? If not, explain how this changes the positioning of the axes on the graph that you drew.
8. Is there a functional relationship between the height of the ball from the ground and time over three tosses of the ball?
9. Sketch a possible graph for three tosses of the ball.
10. Describe the domain and range for three tosses of the ball.
11. Is the height of the ball above the ground a function of its distance from Alexa for a single toss? Explain your reasoning.
12. If distance from Alexa is the independent variable, sketch a possible graph for one toss.
13. What would be a reasonable domain and range?
14. Is the height of the ball above the ground a function of the distance from Alexa for three tosses if each time the two players *do not* toss the ball exactly the same way? Explain.

Notes

Materials:

None required.

Algebra II TEKS Focus:

(2A.1) Foundations for functions. The student uses properties and attributes of functions and applies functions to problem situations.

The student is expected to:

(A) identify the mathematical domains and ranges of functions and determine reasonable domain and range values for continuous and discrete situations.

Additional Algebra II TEKS:
None

Scaffolding Questions:

- If you and a friend are tossing a ball back and forth, does its path follow a straight line?
- Is there a difference between saying a ball is *tossed* and a ball is *thrown*? If so, what is the difference?
- Would there be any difference in the path of the ball if you tossed it underhand or overhand?
- When you play catch do you usually release the ball and catch it at the same height?

Sample Solutions:

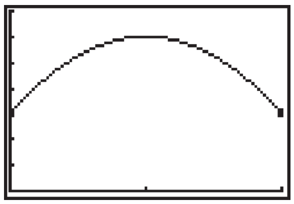
1. Yes, at any time the ball can be only one height.
2. The height of the ball depends on the time that has elapsed since the ball was tossed. The independent variable is time in seconds, and the dependent variable is the height.
3. The ball would go up and then come down as time passes, so the graph might look as follows:

4. The domain is the time in seconds from 0 to the number of seconds it takes to toss the ball one time. The problem states that it takes 2 seconds for one toss. The domain would be $0 \leq x \leq 2$.

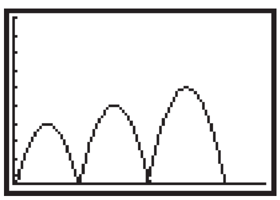
   If the maximum height the ball is tossed is 6 feet and the ball is tossed and caught from 3 feet off the ground, the range would be $3 \leq y \leq 6$

   This assumes the ball is approximately 3 feet from the ground when it is at its lowest point and goes no higher than 6 feet at its highest. Other reasonable ranges are possible.

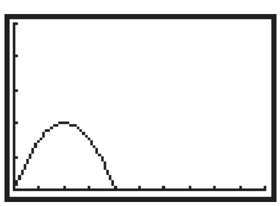
5. The ball started on the ground.
6. At some point the ball was on the ground. Maybe someone dropped it.
7. If a student's original graph is as shown in #3, the axes would have to be shifted to the right and up to match the graph for this problem:



8. The height would be the function of time since the ball was tossed.
9. The graph might look as follows:



10. The domain would be from 0 to 6 seconds (the time it takes to toss the ball 3 times). The range would be from the height from which it was tossed to the maximum height it reaches. The graph above shows the ball tossed from the ground level.
11. Yes, on a single toss the ball would be only at one height for each distance from Alexa.
12. One possible graph is shown below:



Connection to TAKS:

Objective 1: The student will describe functional relationships in a variety of ways.

Objective 2: The student will demonstrate an understanding of the properties and attributes of functions.

Objective 10: The student will demonstrate an understanding of the mathematical processes and tools used in problem solving.

13. Because the distance between the two people is 10 feet, the domain would be $0 \leq x \leq 10$ (in feet).

    Assuming the ball is tossed from 3 feet and the maximum height is 6 feet, the range would be $3 \leq y \leq 6$ (in feet).

14. No. For a functional relationship to exist each time the ball was a given distance from Alexa, it would have to be the exact same height. If it is not tossed the exact same way, this would not necessarily happen. Given the distance, the ball can be on different heights, which means there are different values of *y* for one value of *x*. Therefore, the relationship is not a function.

Extension Questions:

- What parent function would probably be a reasonable model for the height of the ball with respect to time as the ball went from Alexa to Juan?

  The parent function is the quadratic function, or $y = x^2$.

- If a parabola was the best model for the ball as it traveled from Alexa to Juan one time and the maximum height occurred exactly halfway through its flight, describe the relationship between how high the ball was when Alexa tossed it and when Juan caught it.

  If the maximum height was halfway between the two, then Alexa must have released it at the same height that Juan caught it, because the parabola is symmetric about its axis of symmetry.

- Suppose the ball reaches its maximum height much closer to Juan than it is to Alexa when she throws it to him. Does he catch it higher or lower then the height of the ball when Alexa released it?

  Probably higher. The function that models the flight of the ball is symmetric about the vertical line that passes through its vertex (maximum height). If the vertex is closer to Juan, then the ball would have to be in the air longer for it to be at the same height that it was released.

- If you toss a ball straight up over your head, would a plot of its height with respect to time be a function? Explain.

  Yes. There is a unique height (range variable) for each time (domain variable).

- Suppose in the previous problem the independent variable had been the horizontal distance of the ball from you after you tossed it. If it went straight up and down, would a plot of the ball's height vs. this variable be a function?

  No. To be a function each point in the domain must have a unique point in the range. In this situation, there is only one point in the domain with many range values.

Chapter 2:
Transformations

Data Dilemma

The Livestock Show and Rodeo School Art Program is an annual competition for the city's students. Participants are in grades ranging from kindergarten through 12 and must submit an original art project based on Western culture, history, or heritage. Projects are generally created in the fall and then judged by qualified individuals from the show's School Art Committee. Individual school districts select the top 20 students to compete in this annual citywide competition.

The scores for the top Rodeo School Art Competition entries in the high school division in East District are listed below.

Entry	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
Score	65	71	74	77	79	80	81	82	83	84	85	86	87	89	90	90	91	92	93	95

The Art Committee guidelines state that the top rating awarded in district competitions should be 100. The East District judges have decided to add 5 points to each score in order to comply with the competition guidelines.

1. Create a table to show the new scores. Compare the mean and median of the original scores with those of the modified scores.
2. The judges want to see a visual representation of the scores. Graph the original scores using the entry number as the *x*-coordinate and the score as the *y*-coordinate. Describe the parent function to which this graph belongs.
3. Predict how the graph of the modified scores would compare to the graph of the original scores. Graph the modified scores on the same graph as the original scores and check your predictions.
4. If $y = f(x)$ represents the function rule for the original scores, determine a representation for the function rule for the modified scores. Explain your answer.

Notes

Materials:

Graphing calculator

Algebra II TEKS Focus: (2A.4) Algebra and geometry. The student connects algebraic and geometric representations of functions.

The student is expected to:

(A) identify and sketch graphs of parent functions, including linear ($f(x)=x$), quadratic ($f(x)=x^2$), exponential ($f(x)=a^x$) and logarithmic ($f(x)=\log_a x$) functions, absolute value of x ($f(x)=|x|$), square root of x ($f(x)=\sqrt{x}$) and reciprocal of x ($f(x) = 1/x$).

(B) extend parent functions with parameters such as a in $f(x) = a/x$ and describe the effects of the parameter changes on the graph of parent functions.

Additional Algebra II TEKS:

None

Scaffolding Questions:

- What do you have to do with the data to create a table showing the new scores?
- How can you find the mean for this set of data?
- How can you determine the median for this set of data?

Sample Solutions:

1. The table below was created using a graphing calculator. The first list is the entry number. The second list is the given score. The third list is created by adding 5 to the second list value.

L1	L2	L3
1	65	70
2	71	76
3	74	79
4	77	82
5	79	84
6	80	85
7	81	86

L3(1)=70

L1	L2	L3
8	82	87
9	83	88
10	84	89
11	85	90
12	86	91
13	87	92
14	89	94

L3(14)=94

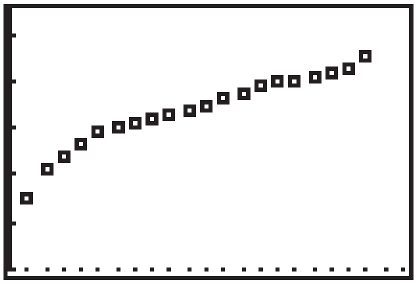
L1	L2	L3
15	90	95
16	90	95
17	91	96
18	92	97
19	93	98
20	95	100
------	------	------

L3(21)=

Finding the sum of the original scores and dividing by 20 yields the mean. The sum of the original scores is 1,674, and the mean is 83.7. The sum of the modified scores is 1,774, and the mean is 88.7. The mean of the modified scores is 5 points higher than the mean of the original scores.

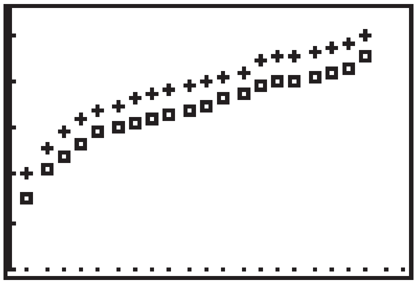
Because the scores are in numerical order, the median is the average of the tenth and eleventh scores in the list. The median of the original scores is the average of 84 and 85 or 84.5. The median of the modified scores is the average of the tenth and eleventh scores on the modified list. The average of 89 and 90 is 89.5. The median of the set of modified scores is 5 points higher than the median of the original set of scores.

2. The scatterplot is shown below.



The parent function is $y = \sqrt{x}$.

3. Since 5 points were added to each score, the graph of the modified scores should be a vertical translation of 5 units of the graph of the original scores.



The graph of the modified scores has the same shape as the original graph, and is shifted up 5 units.

4. The function rule of the modified scores is $y = f(x) + 5$ because the graph is translated up 5 units.

Connection to TAKS:

Objective 1: The student will describe functional relationships in a variety of ways.

Objective 2: The student will demonstrate an understanding of the properties and attributes of functions.

Objective 5: The student will demonstrate an understanding of quadratic and other nonlinear functions.

Objective 9: The student will demonstrate an understanding of percents, proportional relationships, probability, and statistics in application problems.

Objective 10: The student will demonstrate an understanding of the mathematical processes and tools used in problem solving.

Extension Questions:

- Explain algebraically why the mean shown for the modified list is 5 points higher than the mean for the original list.

The sum of the scores of the original list may be represented by $\sum_{i=1}^{20} s_i$ *where* s_i *is each score added. Because there are 20 scores the mean is represented by*

$$\frac{1}{20}\sum_{i=1}^{20} s_i$$

When 5 is added to each score, the sum of the scores is represented by

$$\sum_{i=1}^{20}(s_i + 5)$$

The mean is

$$\frac{1}{20}\sum_{i=1}^{20}(s_i+5) = \frac{1}{20}\sum_{i=1}^{20}(s_i) + \frac{1}{20}\sum_{i=1}^{20}(5) = \frac{1}{20}\sum_{i=1}^{20}(s_i) + \frac{1}{20}20(5) = \frac{1}{20}\sum_{i=1}^{20}(s_i) + (5)$$

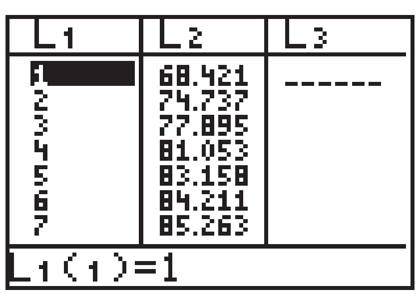
The new mean is the original mean plus 5.

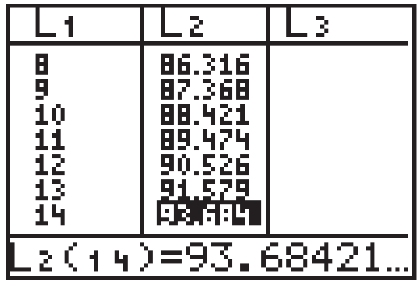
- One of the judges was uncomfortable with the method of modifying the scores by adding 5 points to each score. Instead, he suggested that perhaps the scores should have been multiplied by a factor that would make the highest scores equal 100 points. The judges decided to try this method and compare results.

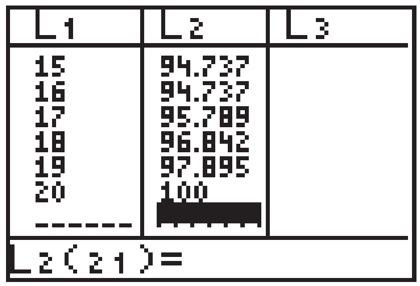
  What factor should be used to multiply the highest original score to obtain 100? Justify your answer.

The highest original score was a 95. Multiplying the number by $\frac{100}{95}$ *will produce a score of 100.*

- Create a table showing the new scores. How does the mean of the original scores compare with that of the second modification of the scores? Explain any differences you find between the comparison of the original scores and the first modification, and the comparison of the original scores and the second modification.







The original scores in L2 were each multiplied by $\frac{100}{95}$ *to produce the newly modified scores.*

The mean of the second modification of the scores is found by adding all of the scores (1,762.105263), and dividing the sum by 20. The mean of the second modification is 88.1 (rounded to the nearest tenth).

The scores on the second modification are "stretched" by a factor of $\frac{100}{95}$*. All of the scores increase, so the mean increases. However, the increase does not follow the same pattern as in the first modification. The first modification involves all scores being raised by the same value; the mean increases by the exact same value also. In the second modification, the scores are multiplied by a factor of* $\frac{100}{95}$*. This results in the new mean being* $\frac{100}{95}$ *times the original mean. This can be shown algebraically.*

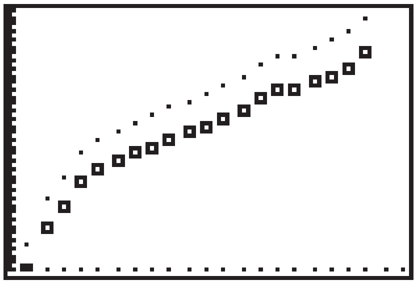
The sum of the scores of the original list may be represented by $\sum_{i=1}^{20} s_i$*. The mean is represented by* $\frac{1}{20}\sum_{i=1}^{20} s_i$*. The mean of the scores multiplied by* $\frac{100}{95}$ *is*

$$\frac{1}{20}\sum_{i=1}^{20}\frac{100}{95}s_i = \frac{1}{20}\bullet\frac{100}{95}\sum_{i=1}^{20}s_i = \frac{100}{95}\bullet\left(\frac{1}{20}\sum_{i=1}^{20}s_i\right).$$

The new mean of the scores is the original mean multiplied by $\frac{100}{95}$*.*

- Graph the newly modified scores on the same graph with the original scores. Describe the change in the graph from the original to the newly modified scores.

The original scores are represented by squares. Dots represent the final modification produced by multiplying each score by $\frac{100}{95}$*. The spread between the scores on the upper end of the graph is visually evident as being bigger than the spread on the lower end of the two graphs.*



- Which method do you think the judges should use to meet the competition guideline of rating the highest scoring artwork a 100? Justify your reasoning.

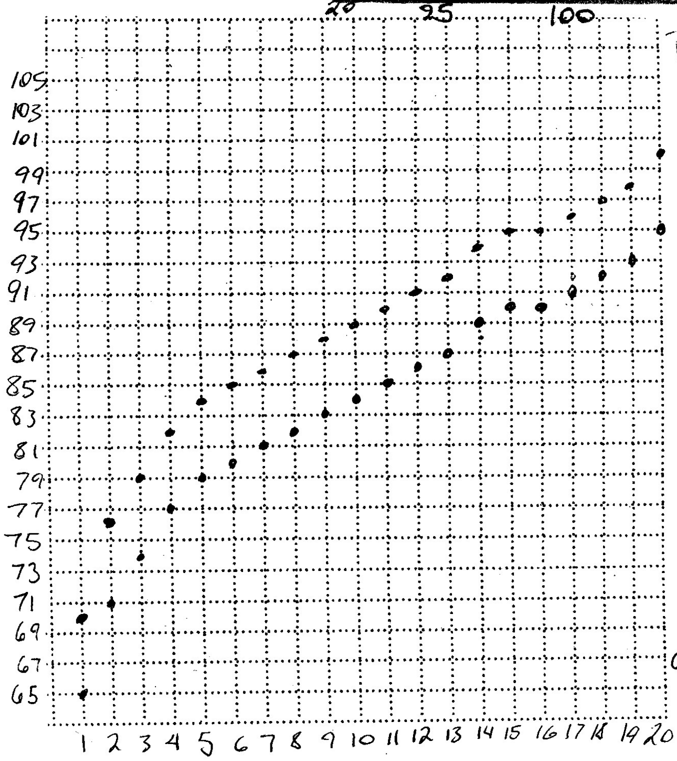
Answers will vary. For example, one might say that the method of adding 5 points is more fair because it helps each student by adding the same number of points to each score. Another student might say that the amount added should vary. More points should be added to the lower scores.

The next four pages of student work show two different approaches to the solution of this problem. The first student solved the problem in a manner similar to the sample solution. The second student decided to try different regression equations and drew a conclusion based entirely on the regression coeffiecient *r*.

Name of Problem Data Dilemma

Table

Entry	Score	New Score
1	65	70
2	71	76
3	74	79
4	77	82
5	79	84
6	80	85
7	81	86
8	82	87
9	83	88
10	84	89
11	85	90
12	86	91
13	87	92
14	89	94
15	90	95
16	90	95
17	91	96
18	92	97
19	93	98
20	95	100



1.) Mean Original Score = 83.7
Mean New Score = 88.7
— I know this is correct b/c the new mean is exactly 5 points greater than the original mean.

(#1 continued next page)

2.) This graph could be either square root or linear.

(Problem continued on next page) →

3.) The graph with the modified scores varies from the graph of the original scores in the following; while the original graph follows f(x), the new graph follows f(x)+5, which results from the additional five points added to each original score.

Name of Problem Data Dilemma

2.) The graphs drawn show both square root and linear tendencies. In a linear perspective, the plotted points seem increase at a constant rate creating a line. However, at the end of the line the plotted points curve down showing similarities to a square root graph.

1.) The mean (average) was found by adding all of the numbers togther then dividing by 20 (amount of numbers added together. This process was performed for both the original and new score.

Name of Problem Data Dilemma

New Scores

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
70	76	79	82	84	85	86	87	88	89	90	91	92	94	95	95	96	97	98	100

Average of original scores → all scores add to 1673. $\frac{1673}{20} = 83.65$

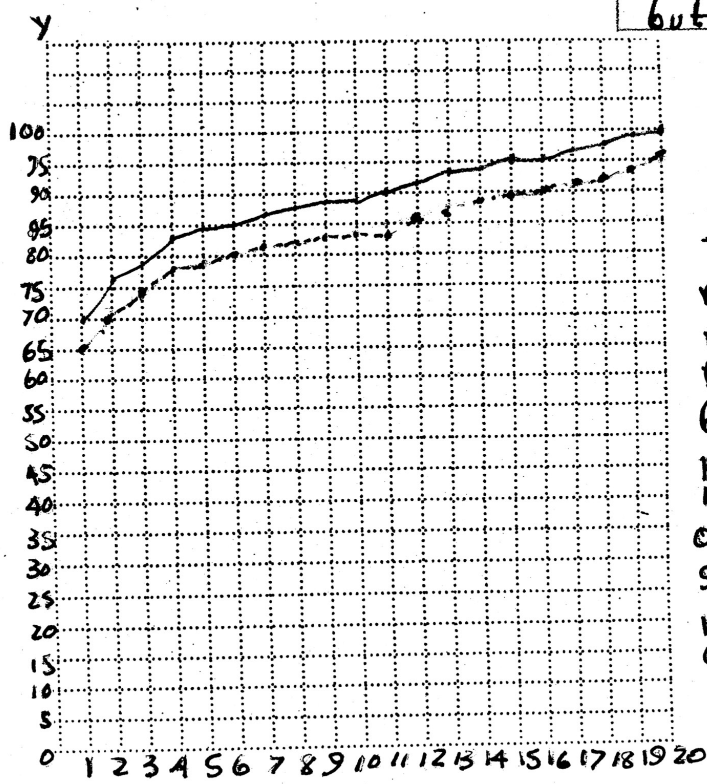
Average of new scores → adds to 1773 $\frac{1773}{20} = 88.65$

the average of the new scores is 5 points higher than the old scores

Green = Original scores

Red = New scores

The graph of the modified scores is 5 units higher on the y-axis than the graph of the original scores but their shape is identical



$y = f(x) + 5$ → This is the equation of the graph, because the table shows the function. The scores relate to the respective students to create the shape, which is the function. The graph described by the formula is identical in shape so the function is the same with 5 units on the y-axis simply added so the formula is described in terms of y since f(x) comes from the original graph

Data and Calculations

Averages

Graph 1 Original $65+71+74+77+72+80+81+82+83+84+85+86+87+89+90+90+91+92+93+95=1{,}673$ $\frac{1673}{20}=83.65$

graph 2 Modified $70+76+79+82+84+85+86+87+88+89+90+91+98+94+95+95+96+97+98+106=1{,}773$ $\frac{1773}{20}=88.65$

Determining the Family of Functions

Linear (LinReg)
$y=ax+b$
$r^2=0.936$
$r=0.967$

Quartic (QuartReg)
$y=ax^4+bx^3+cx^2+dx+e$
$r^2=0.595$

Quadratic (QuadReg)
$y=ax^2+bx+c$
$r=0.969$

Exponential (ExpReg)
$y=a(b^x)$
$r^2=0.952$
$r=0.952$

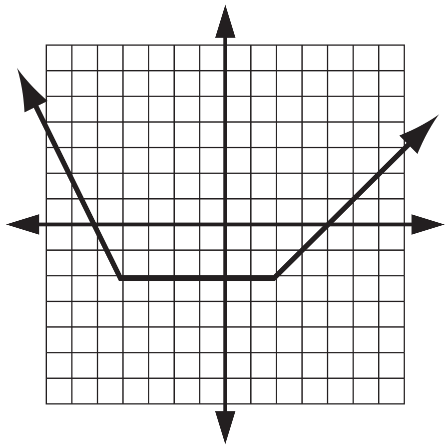
Cubic (CubicReg)
$y=ax^3+bx^2+cx+d$
$r^2=0.989$

Power (PwrReg)
$y=a(x^2)$
$r^2=0.989$
$r=0.994$

The closest r^2 to 1 is 0.995, which I get for Quartic I have therefore determined it is Quartic.

Slip Sliding Away

The graph of the function *y* = *f(x)* is given below.



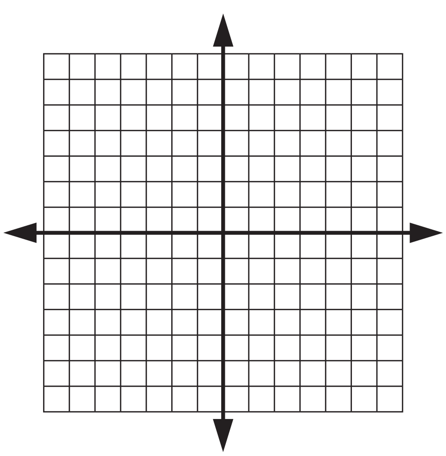
For each of the following problems,

a. Draw each requested transformation, or combination of transformations of the given function on a separate graph.

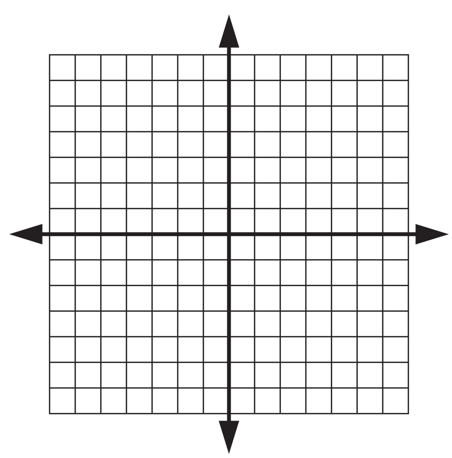
b. Describe the effect of the transformation on the parent function.

c. Describe the range and the domain of the transformed function.

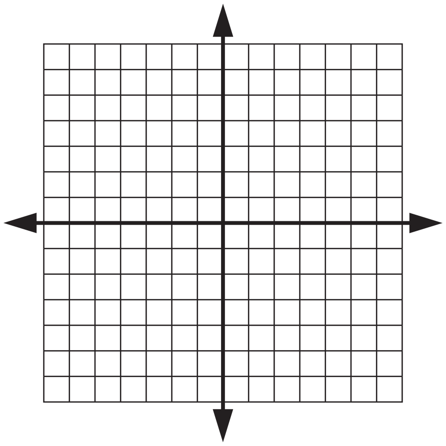
1. *f(x)* – 2



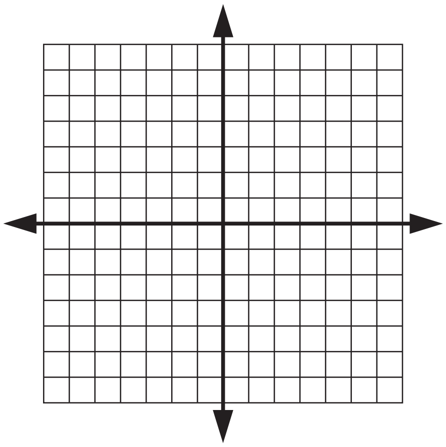
2. -*f(x)*



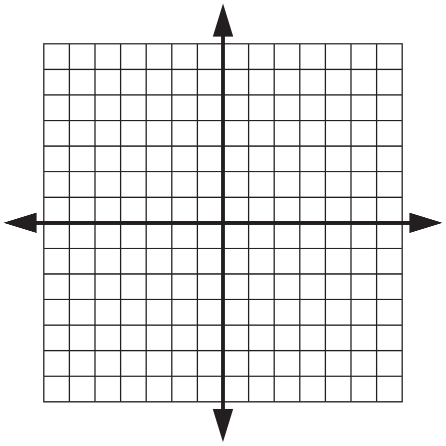
3. $f(-x) + 1$



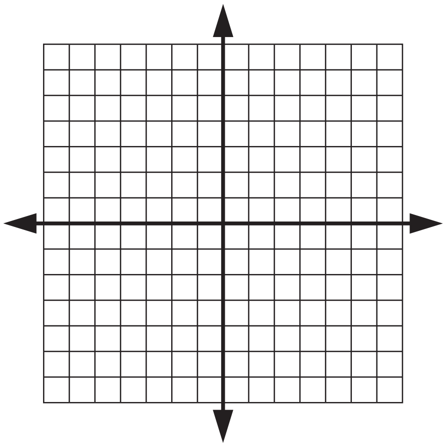
4. $f(x + 3)$



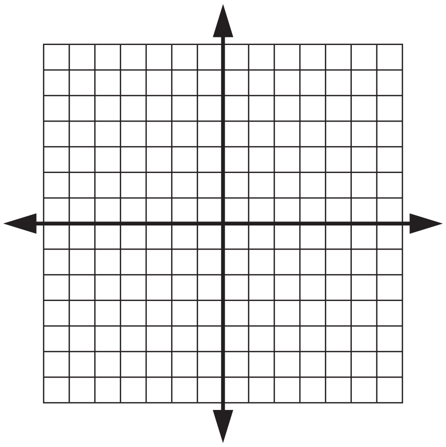
5. $f(x - 2) + 1$



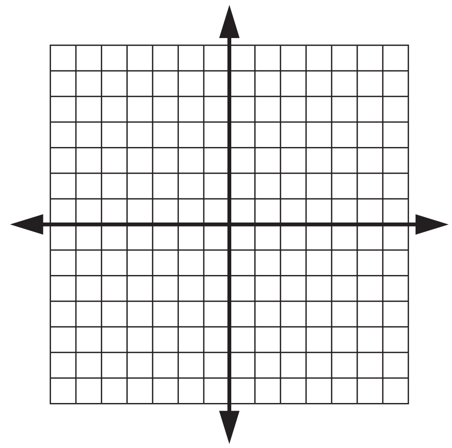
6. $2f(x)$



7. $\frac{1}{2}f(x)+3$



8. $3f(x-2)+1$



Notes

Materials:

Graphing calculator

Algebra II TEKS Focus:
(2A.4) Algebra and geometry. The student connects algebraic and geometric representations of functions.

The student is expected to:

(B) extend parent functions with parameters such as a in and $f(x) = a/x$ describe the effects of the parameter changes on the graph of parent functions.

Additional Algebra II TEKS:
(2A.1) Foundations for functions. The student uses properties and attributes of functions and applies functions to problem situations.

The student is expected to:

(A) identify the mathematical domains and ranges of functions and determine reasonable domain and range values for continuous and discrete situations.

Scaffolding Questions:

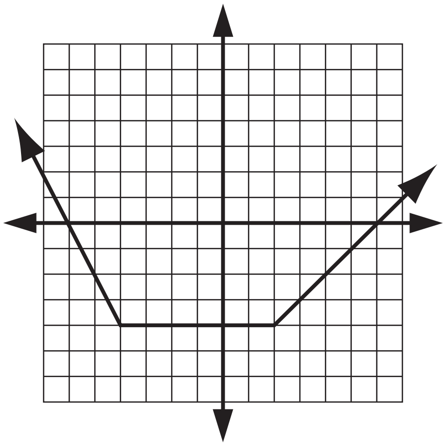
- How would the function $f(x) = x$ be modified to show a vertical translation of 2 units up?
- How would the function $f(x) = x^2$ be modified to show a translation 5 units to the right?
- How would you modify the function $f(x) = x^2$ to show a translation of 5 units to the right and a vertical translation of 2 units up?
- Does the order of the actions for translations make a difference?

Sample Solutions:

1. The function $f(x) - 2$ is translated 2 units down.

   The domain is all reals.

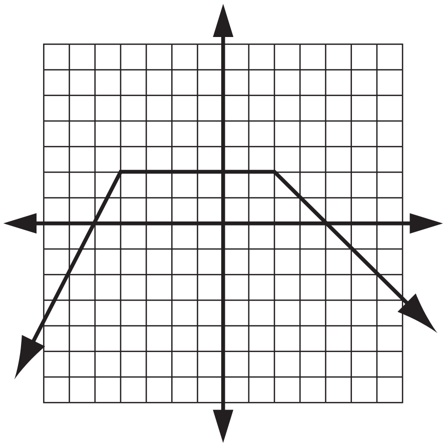
   The range is $y \geq -4$.



2. The function -*f*(*x*) is reflected over the *x*-axis.

   The domain is all reals.

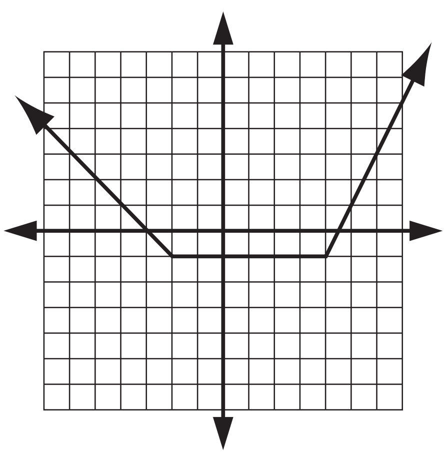
   The range is $y \leq 2$.



3. The function *f*(-*x*) + 1 is reflected over the *y*-axis, and translates up 1 unit.

   The domain is all reals.

   The range is $y \geq -1$.



Connection to TAKS:

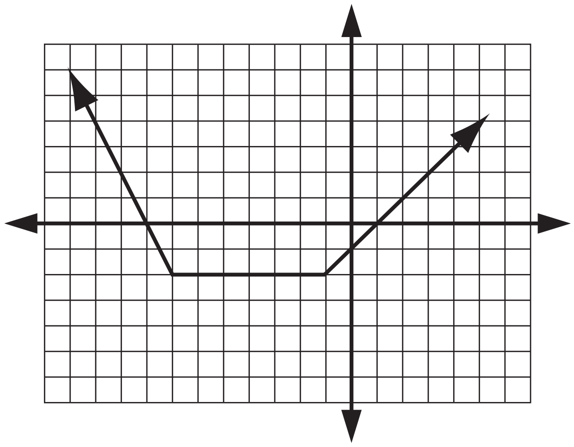
Objective 1: The student will describe functional relationships in a variety of ways.

Objective 2: The student will demonstrate an understanding of the properties and attributes of functions.

Objective 5: The student will demonstrate an understanding of quadratic and other nonlinear functions.

Objective 10: The student will demonstrate an understanding of the mathematical processes and tools used in problem solving.

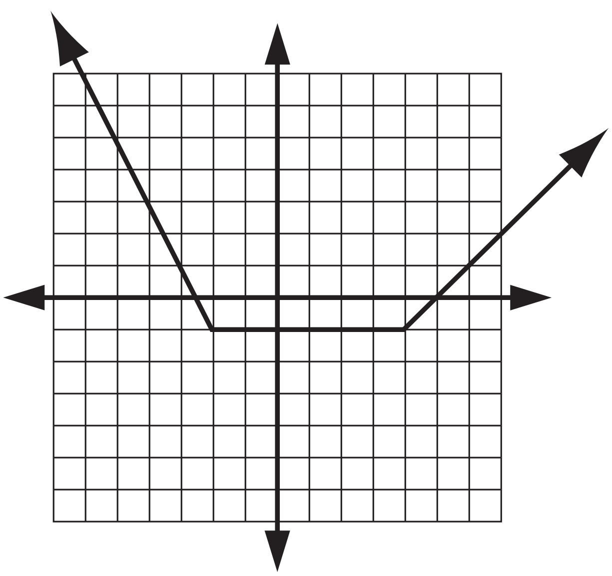
4. The function $f(x + 3)$ is translated to the left 3 units. The domain is all reals. The range is $y \geq -2$.



5. The function $f(x - 2)+1$ is translated to the right 2 units and translated up 1 unit.

   The domain is all reals.

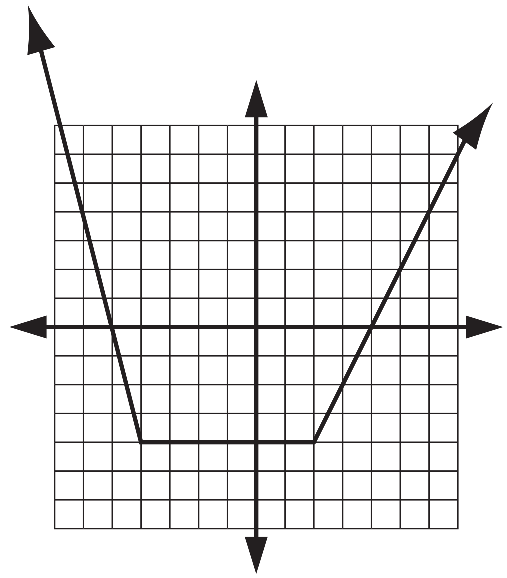
   The range is $y \geq -1$.



6. For the function $2f(x)$ the y-values are multiplied by 2.

   The domain is all reals.

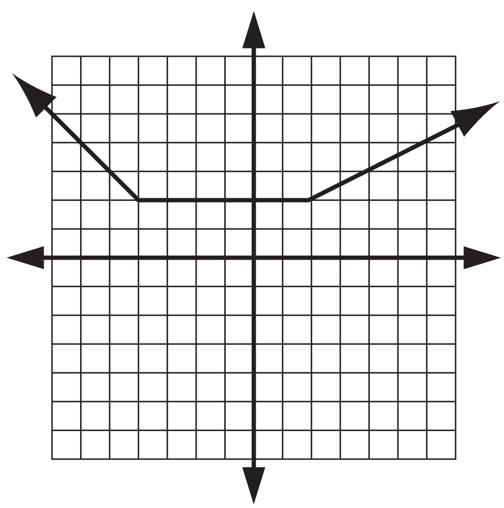
   The range is $y \geq -4$.



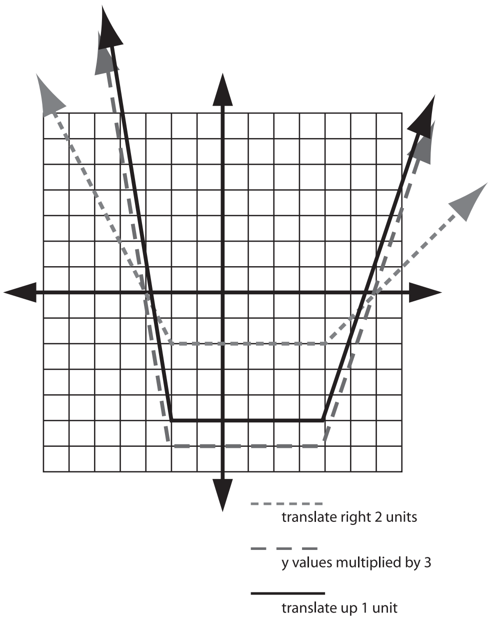
7. For the function $\frac{1}{2}f(x) + 3$ the range values are multiplied by one-half and the graph is translated up 3 units.

   The domain is all reals.

   The range is $y \geq 2$.



8. For the function $3f(x-2)+1$ the original graph is translated to the right 2 units. The *y*-values are multipled by 3. The graph is raised up 1 unit. The range is $y \geq -5$.

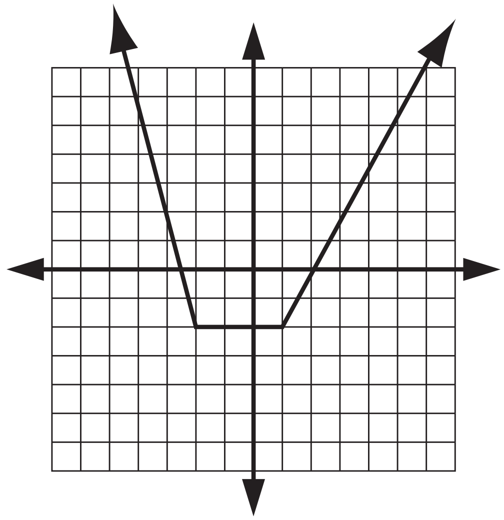


Extension Questions:

- Sketch the graph of the function $f(2x)$.

  A table helps to see the effects of this change on the function. To determine the original values, look at the graph.

x	**2*x***	***f*(2*x*)**
-4	-8	6
-3	-6	2
-2	-4	-2
-1	-2	-2
0	0	-2
1	2	-2
2	4	0
3	6	2
4	8	4



- Sketch an absolute value function with its vertex in the first quadrant. Use a translation to move the vertex into the third quadrant. Describe your translation verbally and symbolically.

  Sample solution: Horizontal and vertical translations can be combined to produce diagonal translations. The function representing the figure would have 6 units added to it in order to translate it left into the second quadrant. The function rule would have 11 units subtracted from it in order to vertically translate down into the third quadrant.

  Function:

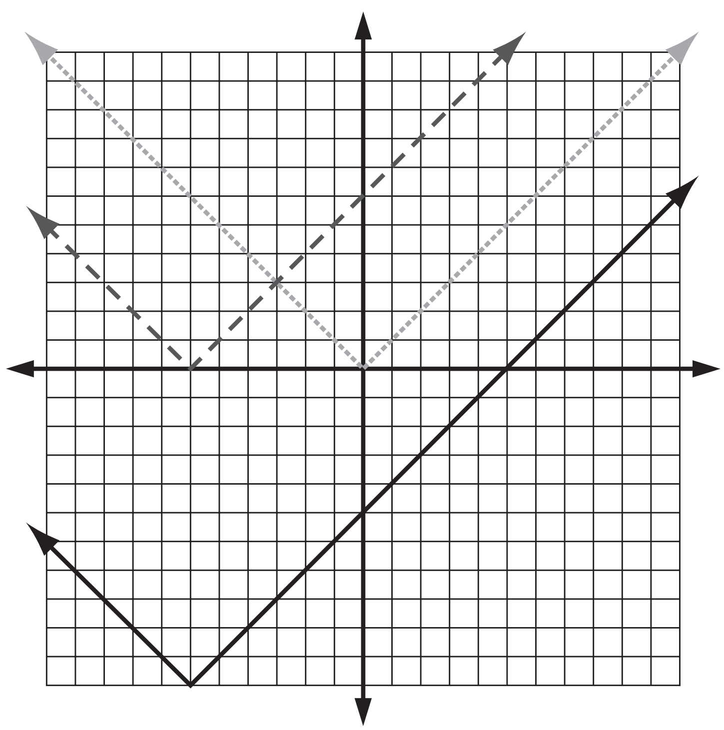
  $y = |x|$

  First translation:

  $y = | x + 6 |$

  Second translation:

  $y = | x + 6 | - 11$



Transformation Two-Step

1. Describe each of the following families of functions in each set.
2. For each set of functions, sketch the graph of the parent function and write its rule for its graph.
3. Graph and label the axes for the transformation.
4. Write a verbal description for each transformation in each set describing its relationship to the parent graph.
5. Describe the domain and range of the function.

$f(x)$	Name of family of functions	Parent function graph and rule	Graph	Description of transformation	Domain of $f(x)$	Range of $f(x)$
Set A						
$f(x) = \lvert x - 3 \rvert$						
$f(x) = -\lvert x \rvert + 1$						
$f(x) = 0.5 \lvert x \rvert$						

$f(x)$	Name of family of functions	Parent function graph and rule	Graph	Description of transformation	Domain of $f(x)$	Range of $f(x)$
Set B						
$f(x) = (x + 6)^2 - 2$						
$f(x) = -4x^2 + 3$						
$f(x) = \frac{1}{3}x^2$						

$f(x)$	Name of family of functions	Parent function graph and rule	Graph	Description of transformation	Domain of $f(x)$	Range of $f(x)$
Set C						
$f(x) = 2^x + 1$						
$f(x) = 2^{x+2}$						
$f(x) = 3(2^x)$						

$f(x)$	Name of family of functions	Parent function graph and rule	Graph	Description of transformation	Domain of $f(x)$	Range of $f(x)$
Set D						
$f(x) = 2\log_2 x$						
$f(x) = \log_2(x - 1)$						
$f(x) = -\log_2 x - 1$						

Notes

Materials:

None required.

Algebra II TEKS Focus:
(2A.4) Algebra and geometry. The student connects algebraic and geometric representations of functions.

The student is expected to:

(A) identify and sketch graphs of parent functions, including linear ($f(x) = x$), quadratic ($f(x) = x^2$), exponential ($f(x) = a^x$), and logarithmic ($f(x) = \log_a x$) functions, absolute value of x ($f(x) = |x|$), square root of x ($f(x) = \sqrt{x}$) and the reciprocal of x ($f(x) = 1/x$).

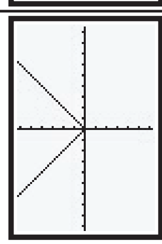
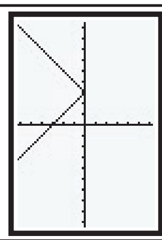
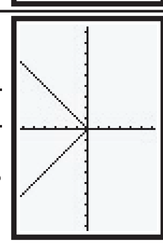
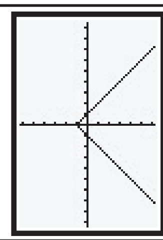
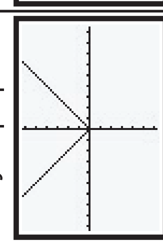
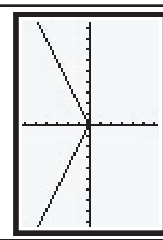
(B) extend parent functions with parameters such as a in $f(x) = a/x$ and describe the effects of the parameter changes on the graph of parent functions.

(C) describe and analyze the relationship between a function and its inverse.

Scaffolding Questions:

- How would you modify the function $f(x) = |x|$ to show that it is shifted 5 units up?
- How would the function $y = x^2$ be modified to show a translation 5 units to the right?
- How would you modify the function $y = x^2$ to show a translation of 5 units to the right and a vertical translation of 2 units up?

Sample Solutions:

$f(x)$	Name of family of functions	Parent function graph and rule	Graph	Description of transformation	Domain of $f(x)$	Range of $f(x)$
Set A						
$f(x) = \lvert x - 3 \rvert$	absolute value	$y = \lvert x \rvert$ 		Shifted 3 units to the right.	all real numbers	$\{y: y \geq 0\}$
$f(x) = -\lvert x \rvert + 1$	absolute value	$y = \lvert x \rvert$ 		Reflected over the y-axis and shifted up 1 unit.	all real numbers	$\{y: y \leq 1\}$
$f(x) = 0.5\lvert x \rvert$	absolute value	$y = \lvert x \rvert$ 		Each y-value is multiplied by one-half.	all real numbers	$\{y: y \geq 0\}$

Additional Algebra II TEKS:
(2A.1) Foundations for functions. The student uses properties and attributes of functions and applies functions to problem situations.

The student is expected to:

(A) identify the mathematical domains and ranges of functions and determine reasonable domain and range values for continuous and discrete situations.

(2A.7) Quadratic and square root functions. The student interprets and describes the effects of changes in the parameters of quadratic functions in applied and mathematical situations.

The student is expected to:

(A) use characteristics of the quadratic parent function to sketch the related graphs and connect between the $y = ax^2 + bx + c$ and the $y = a(x - h)^2 + k$ symbolic representations of quadratic functions.

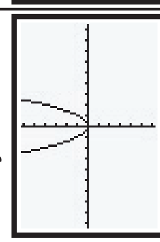
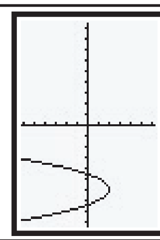
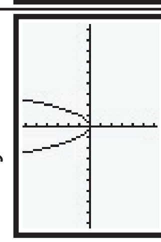
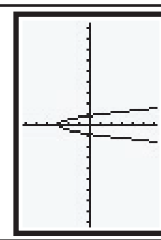
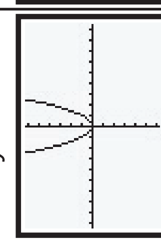
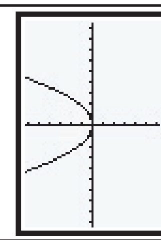
(2A.11) Exponential and logarithmic functions. The student formulates equations and inequalities based on exponential and logarithmic functions, uses a variety of

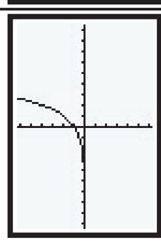
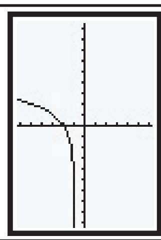
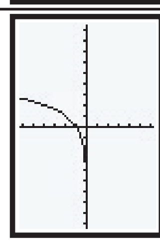
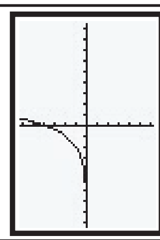
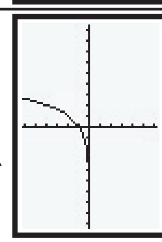
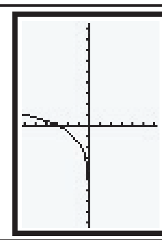
Notes

methods to solve them, and analyzes the solutions in terms of the situation.

The student is expected to:

> (B) use the parent functions to investigate, describe, and predict the effects of parameter changes on the graphs of exponential and logarithmic functions, describe limitations on the domains and ranges, and examine asymptotic behavior.

f (x)	**Name of family of functions**	**Parent function graph and rule**	**Graph**	**Description of transformation**	**Domain of f(x)**	**Range of f(x)**
Set B						
$f(x) = (x + 6)^2 - 2$	quadratic	$y = x^2$ 		Shifted to the left 6 units and shifted down 2 units.	all real numbers	$\{y: y \geq -2\}$
$f(x) = -4x^2 + 3$	quadratic	$y = x^2$ 		Each *y*-value is multiplied by -4 and then the graph is raised up 3 units.	all real numbers	$\{y: y \leq 3\}$
$f(x) = \frac{1}{3}x^2$	quadratic	$y = x^2$ 		Each *y*-value is multiplied by one-third.	all real numbers	$\{y: y \geq 0\}$

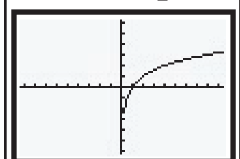
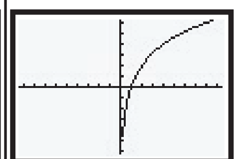
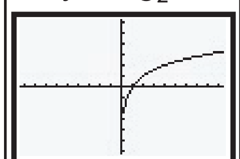
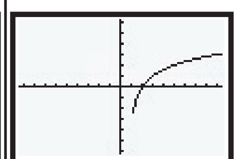
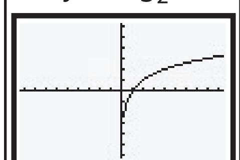
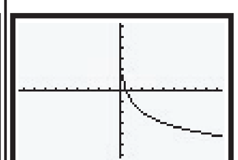
$f(x)$	Name of family of functions	Parent function graph and rule	Graph	Description of transformation	Domain of $f(x)$	Range of $f(x)$
Set C						
$f(x) = 2^x + 1$	base 2 exponential	$y = 2^x$ 		Shifted up 1 unit.	all real numbers	$\{y: y > 1\}$
$f(x) = 2^{x+2}$	base 2 exponential	$y = 2^x$ 		Shifted to the left 2 units.	all real numbers	$\{y: y > 0\}$
$f(x) = 3(2^x)$	base 2 exponential	$y = 2^x$ 		Each y-value is multiplied by 3.	all real numbers	$\{y: y > 0\}$

Connection to TAKS:

Objective 1: The student will describe functional relationships in a variety of ways.

Objective 2: The student will demonstrate an understanding of the properties and attributes of functions.

Objective 5: The student will demonstrate an understanding of quadratic and other nonlinear functions.

f(*x*)	Name of family of functions	Parent function graph and rule	Graph	Description of transformation	Domain of *f*(*x*)	Range of *f*(*x*)
Set D						
$f(x) = 2\log_2 x$	base 2 logarithm	$y = \log_2 x$ 		Each *y*-value is multiplyed by 2.	$\{x: x > 0\}$	all real numbers
$f(x) = \log_2(x-1)$	base 2 logarithm	$y = \log_2 x$ 		Shifted to the left 2 units.	$\{x: x > 1\}$	all real numbers
$f(x) = -\log_2 x - 1$	base 2 logarithm	$y = \log_2 x$ 		The graph is reflected over the *x*-axis and then shifted down 1 unit.	$\{x: x > 0\}$	all real numbers

Extension Questions:

- Are any pairs of the given functions inverses of each other? Explain why.

  The functions $f(x) = \log_2 (x - 1)$ and $g(x) = 2^x + 1$ are inverse functions. If the f(x) is solved for x, the result is in the form of g(x) with x replacing y and y replacing x.

  $$y = \log_2 (x - 1)$$

  $$2^y = x - 1$$

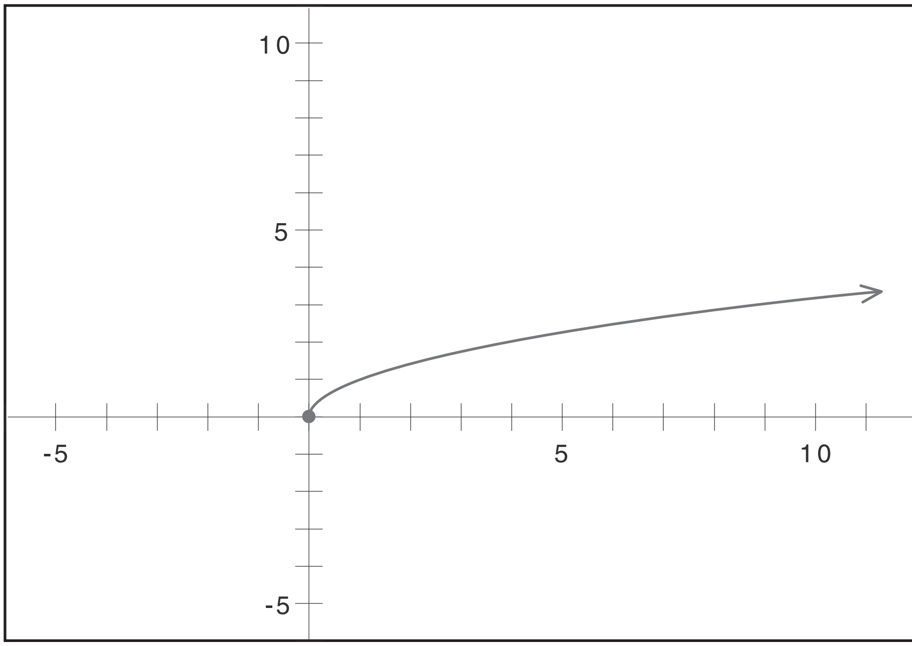
  $$x = 2^y + 1$$

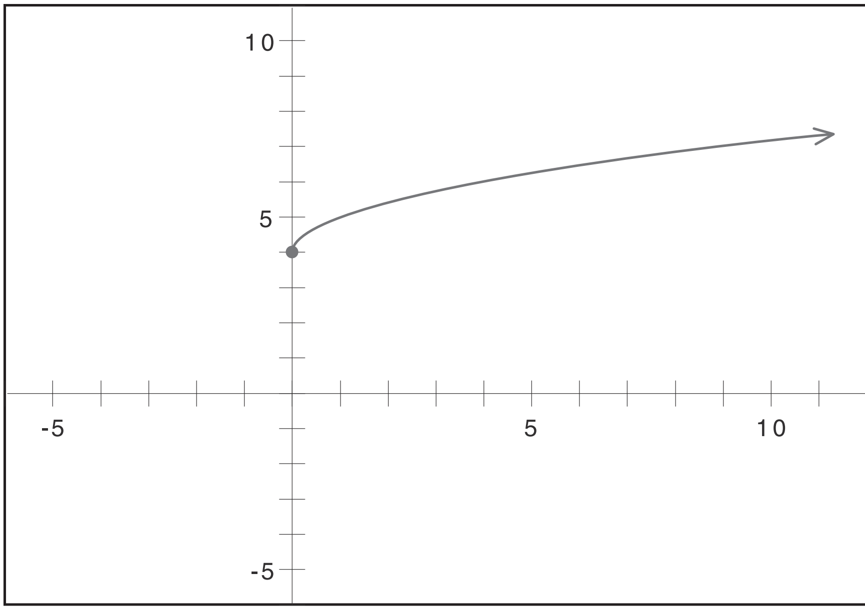
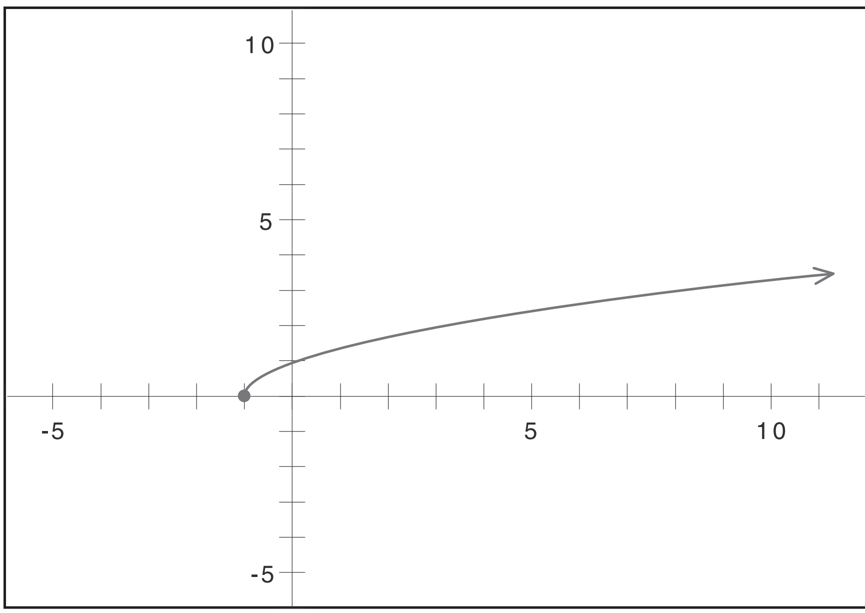
Investigating the Effect of *a*, *h*, and *k* on $y = a\sqrt{x-h}+k$

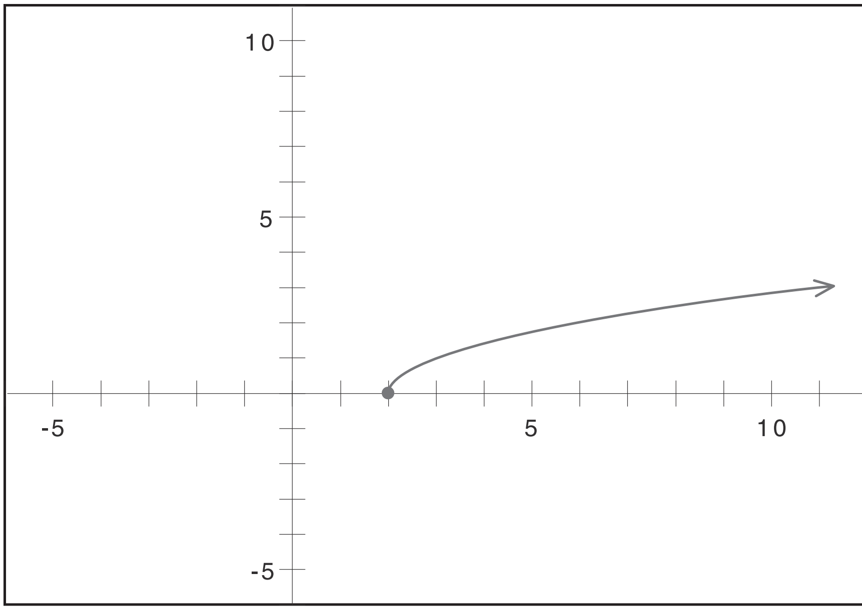
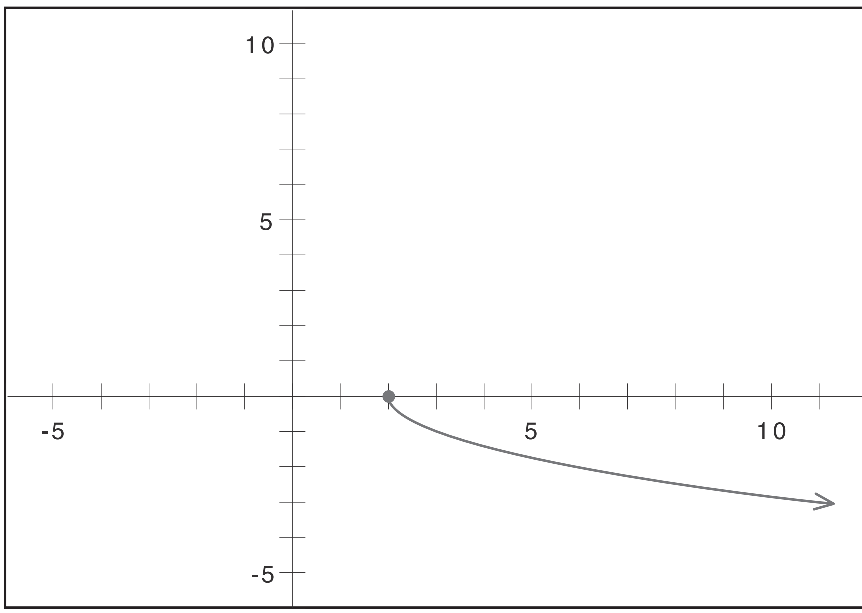
For each of the sets of functions in Activities 1, 2, and 3, complete the tables comparing their graphs with the graph of the parent function $y = \sqrt{x}$.

Activity Worksheet 1

Complete the table to compare the graph of the pictured function with the graph of the parent function, $y = \sqrt{x}$. Describe which parameter of the general function, $y = a\sqrt{x-h}+k$, has been changed and if the new value is a positive or negative number. All windows have the same settings.

	Graph of function	Parameters changed	Changed parameter is positive or negative
1.		Parent function	No change

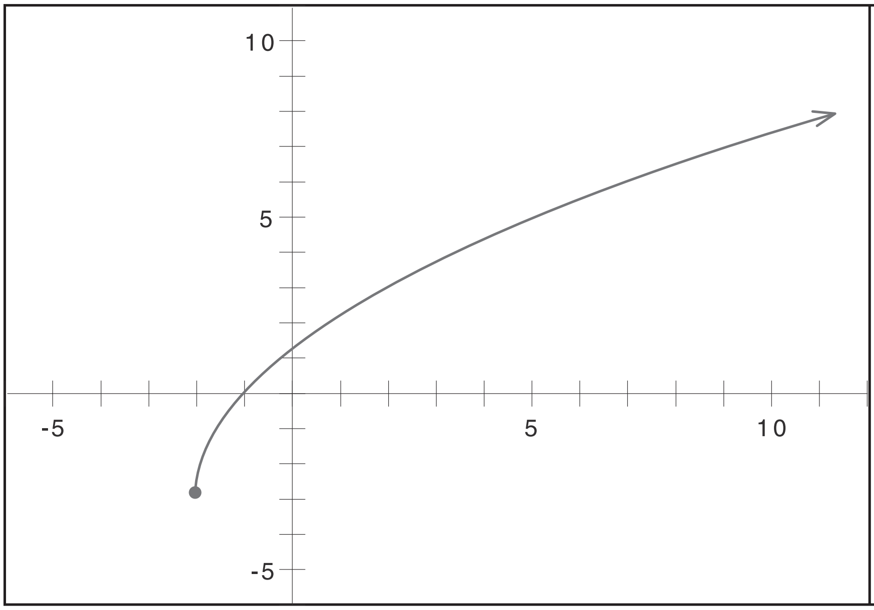
	Graph of function	Parameters changed	Changed parameter is positive or negative
2.			
3.			

	Graph of function	Parameters changed	Changed parameter is positive or negative
4.			
5.			

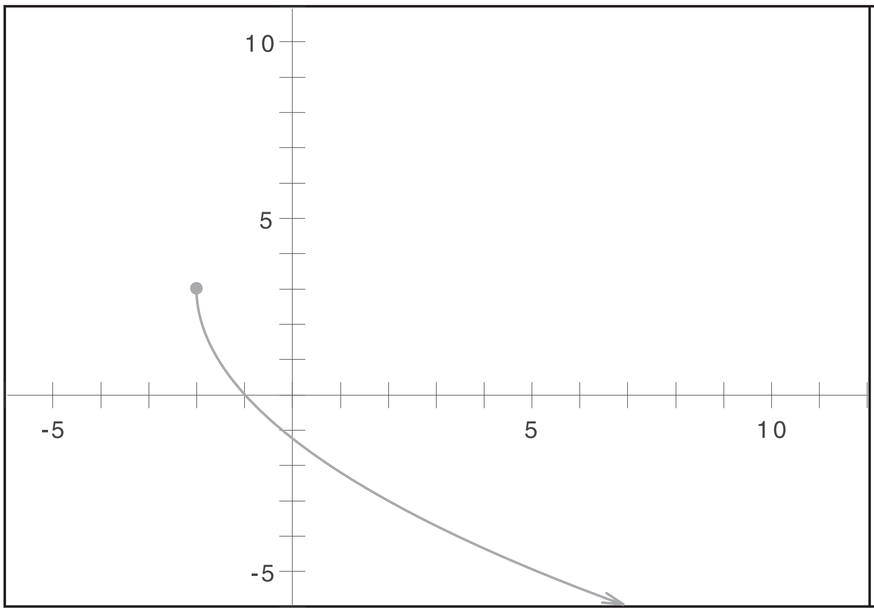
Changed parameter is positive or negative

Graph of function **Parameters changed**

6.

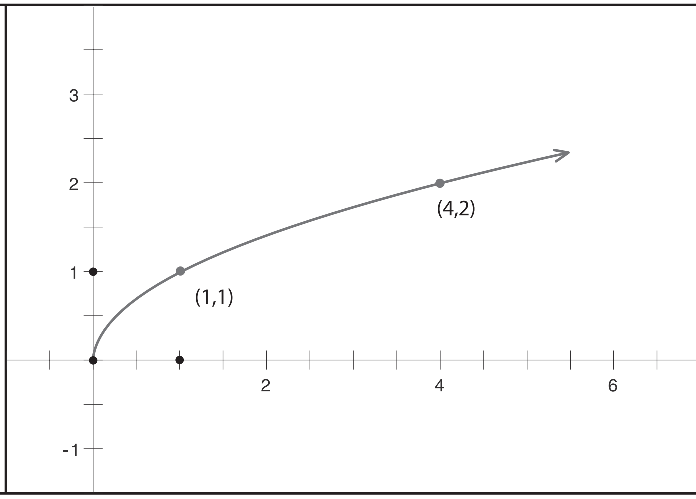
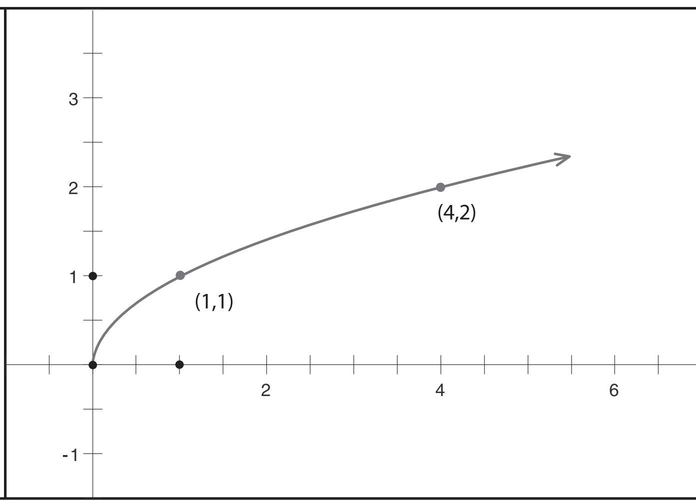


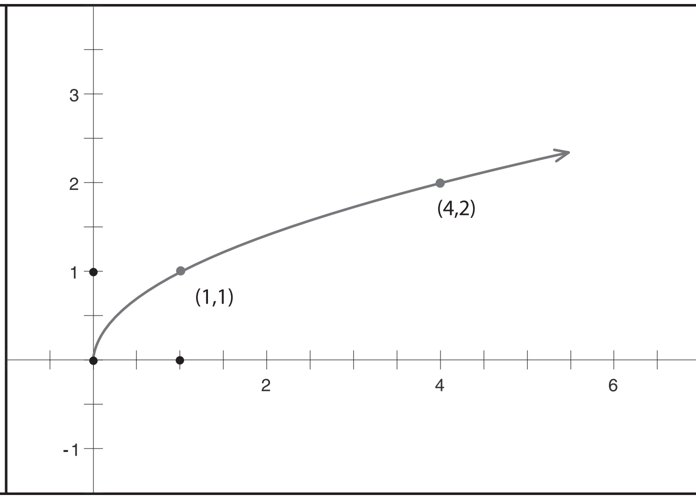
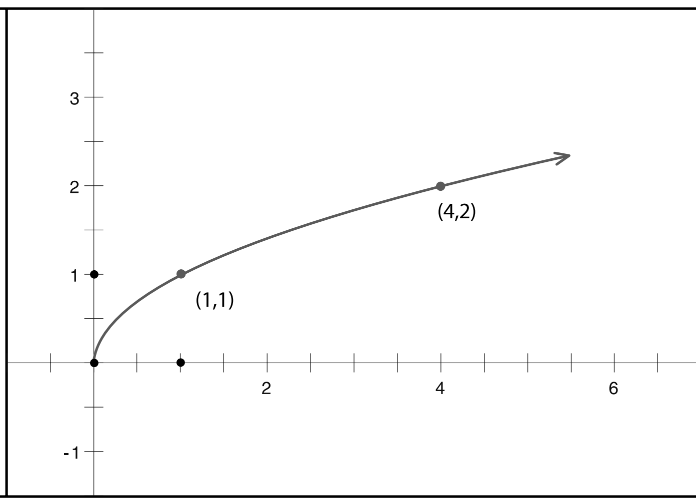
7.

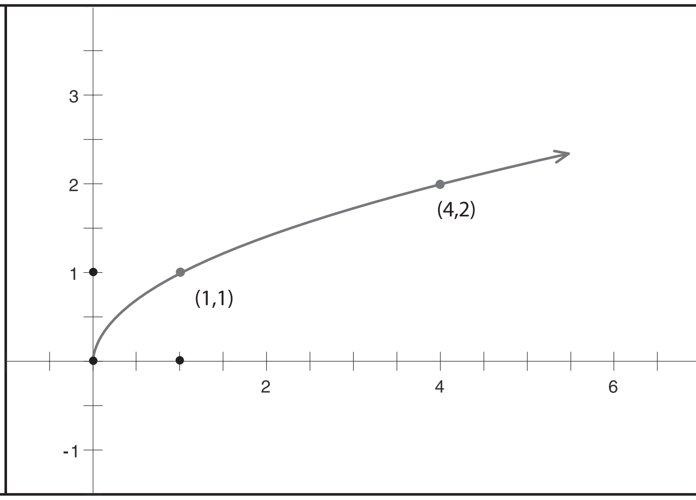
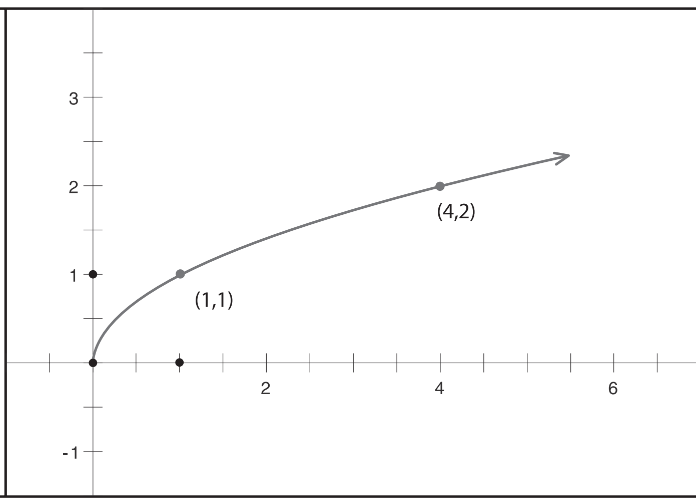


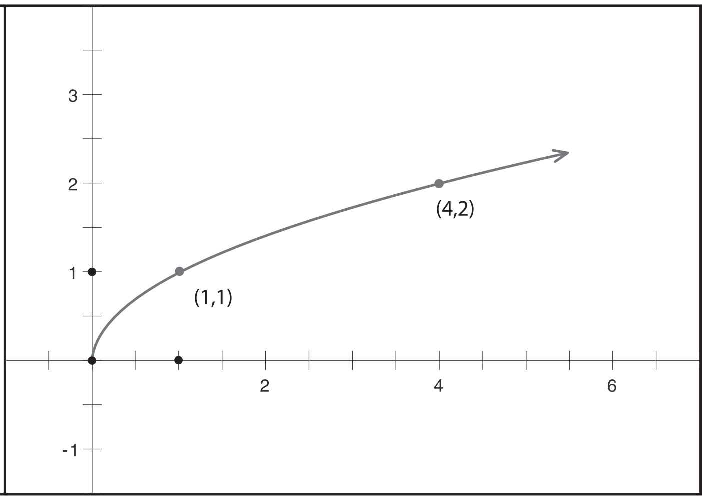
Activity Worksheet 2

Complete the table to compare the graph of the given function with the graph of the parent function, $y = \sqrt{x}$. On the graph show the new points to which the points on the parent graph are translated by the transformation. Describe how changing *a*, *h,* and/or *k* affects the shape and location of the parent function. List the domain and range of the new function.

	Function	Sketch the graph of the listed function on the given coordinate system. The parent graph is given.	Describe any transfor-mation that was used.	List the domain and range
1.	$y = \sqrt{x - 2}$			
2.	$y = \sqrt{x} - 2$			

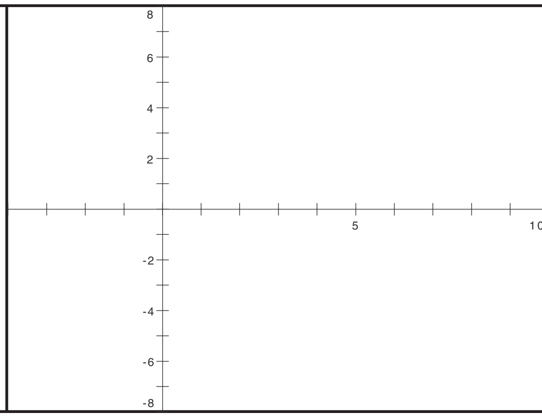
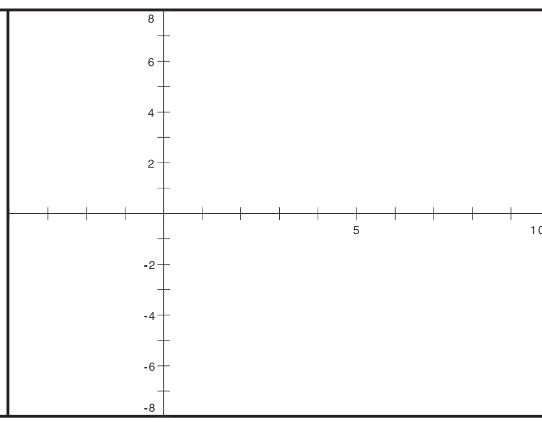
	Function	Sketch the graph of the listed function on the given coordinate system. The parent graph is given.	Describe any transfor-mation that was used.	List the domain and range
3.	$y = -2\sqrt{x}$			
4.	$y = \sqrt{x} + 5$			

	Function	Sketch the graph of the listed function on the given coordinate system. The parent graph is given.	Describe any transfor-mation that was used.	List the domain and range
5.	$y = \sqrt{x+3} - 5$			
6.	$y = 4\sqrt{x} - 2$			

	Function	Sketch the graph of the listed function on the given coordinate system. The parent graph is given.	Describe any transformation that was used.	List the domain and range
7.	$y = 4\sqrt{x-2} + 3$			

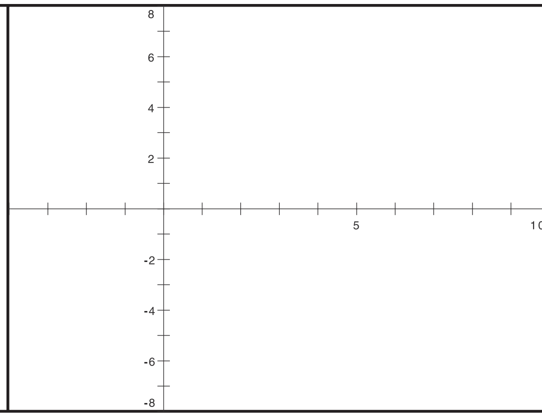
Activity Worksheet 3

Complete the table to compare the graph of the described function with the graph of the parent function, $y = \sqrt{x}$. In each case either a translation will be given or a domain and range will be listed. Sketch the graph of the function of the form $y = \sqrt{x-h} + k$ that is described.

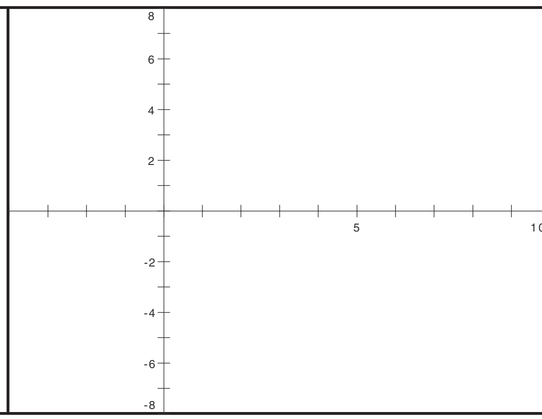
	Description of change to parent function	Graph	Rule of described function
1.	The parent graph is translated 3 units to the right.		
2.	The parent graph is translated 1 unit to the left and 2 units down.		

Description of change to parent function | Graph | Rule of described function

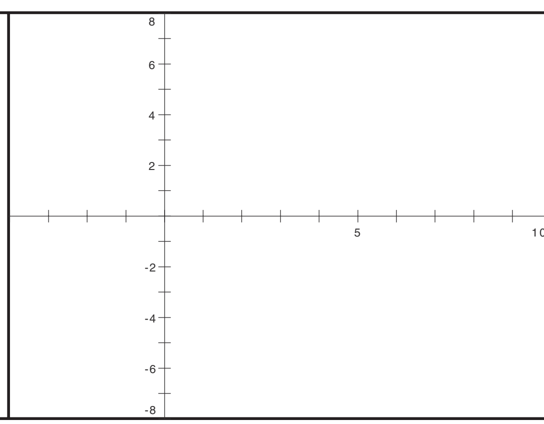
3. The parent graph is translated 1 unit up.



4. The parent graph is translated 3 units to the left and 1 unit up.



5. The parent graph is translated 3 units to the left, reflected across the *x*-axis and translated 4 units down.



Notes

Materials:

None required.

Algebra II TEKS Focus: (2A.9) Quadratic and square root functions. The student formulates equations and inequalities based on square root functions, uses a variety of methods to solve them, and analyzes the solutions in terms of the situation.

The student is expected to:

(A) use the parent function to investigate, describe, and predict the effects of parameter changes on the graphs of square root functions and describe limitations on the domains and ranges.

(B) relate representations of square root functions, such as algebraic, tabular, graphical, and verbal descriptions.

Additional Algebra II TEKS:
None

Scaffolding Questions:

- How does a change in b affect the graph of a line with a rule given in y-intercept form, $y = mx + b$?
- How do the values of h and k affect the graph of a parabola in $y = a(x-h)^2 + k$ form?
- What effect does the value of a have on the parabola?
- What point on the square root function relates to the vertex of the parabola?
- Why does the square root function have a restricted domain and range?
- Does it matter in what order you do the transformations?

Connection to TAKS:

Objective 2: The student will demonstrate an understanding of the properties and attributes of functions.

Objective 5: The student will demonstrate an understanding of quadratic and other nonlinear functions.

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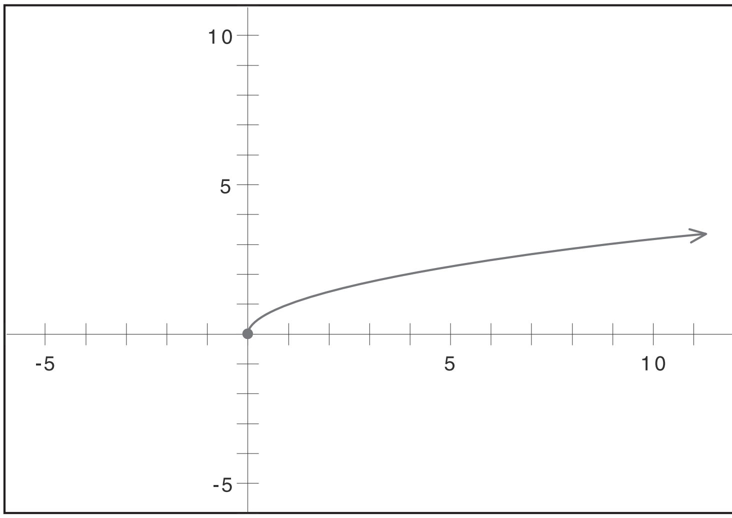
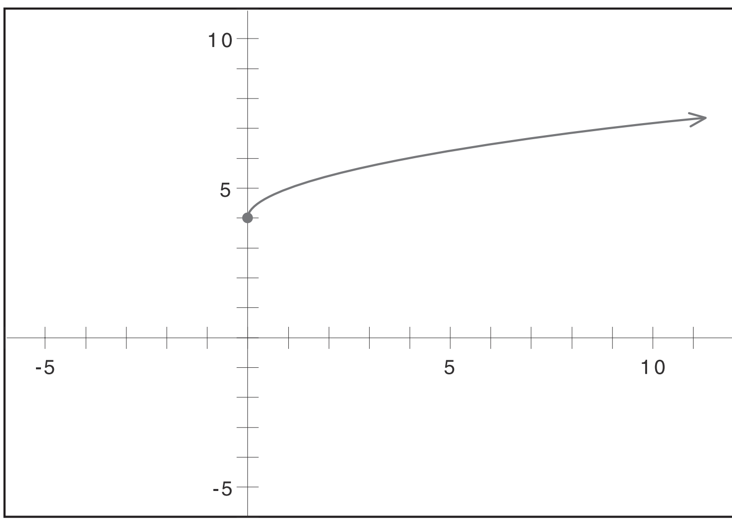
Sample Solutions:

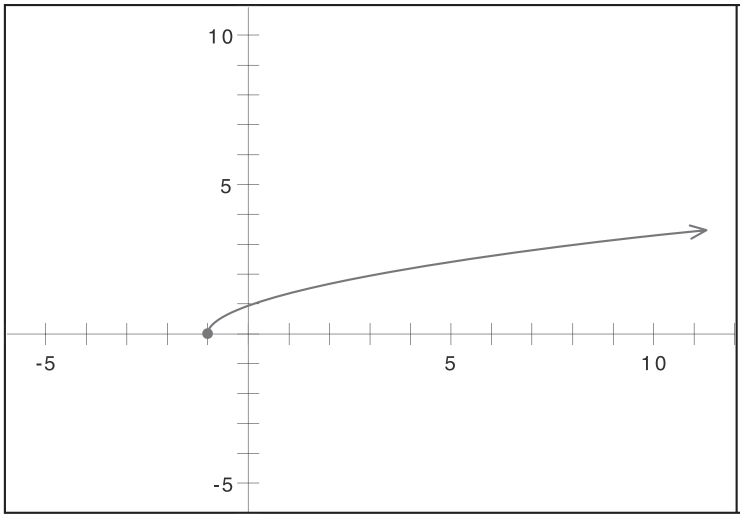
Activity Worksheet 1

	Graph of function	Parameters changed	Changed parameter is positive or negative
1.		Parent function	No change
2.		changed k	$k > 0$

Changed parameter is positive or negative

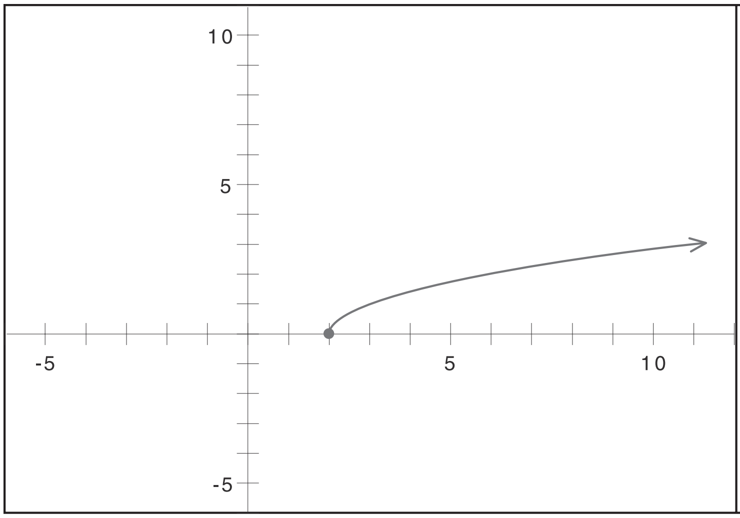
Graph of function **Parameters changed**

3.



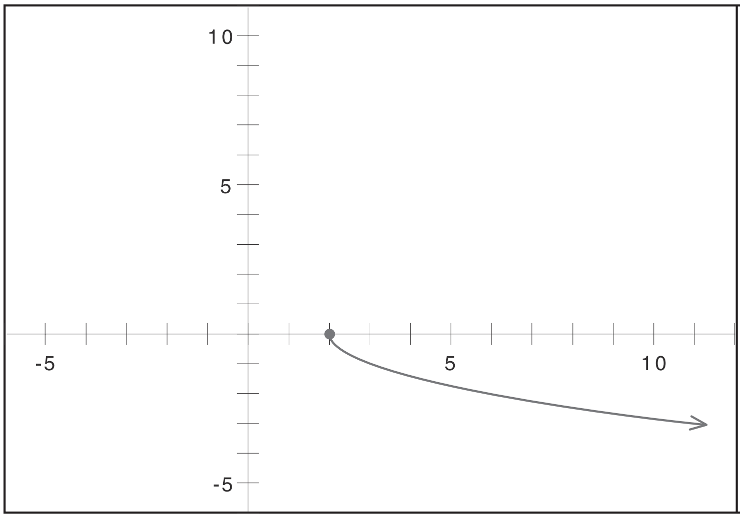
changed h | $h < 0$

4.

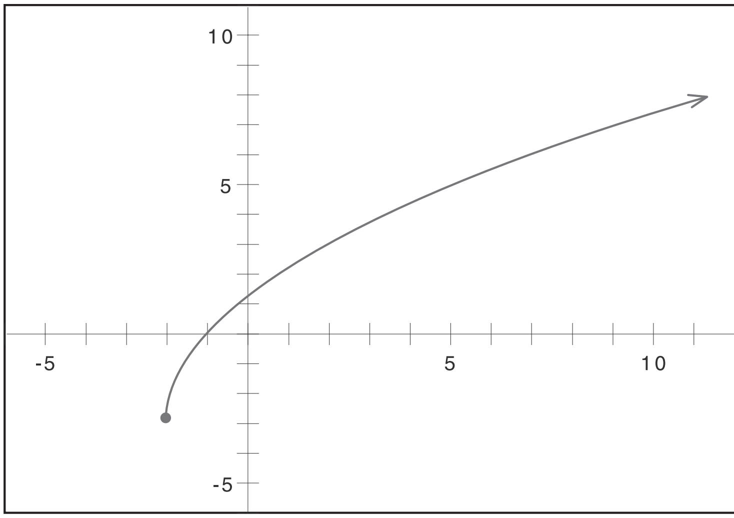
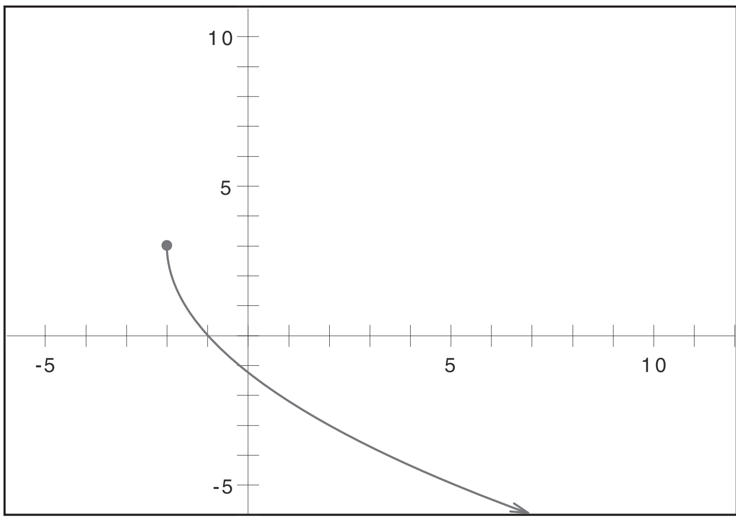


changed h | $h > 0$

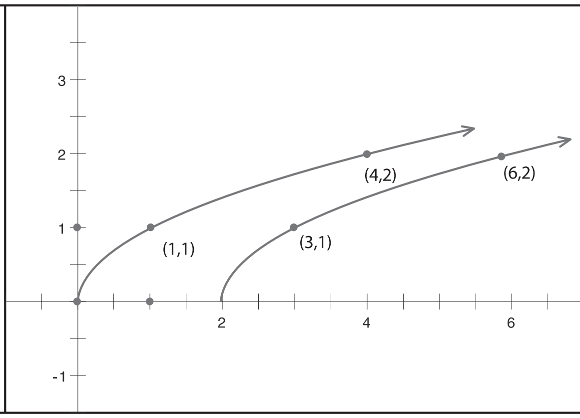
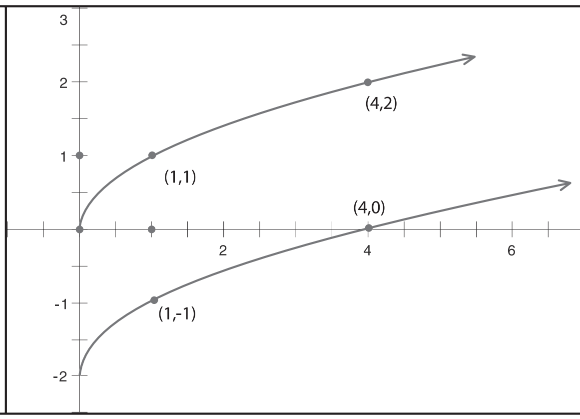
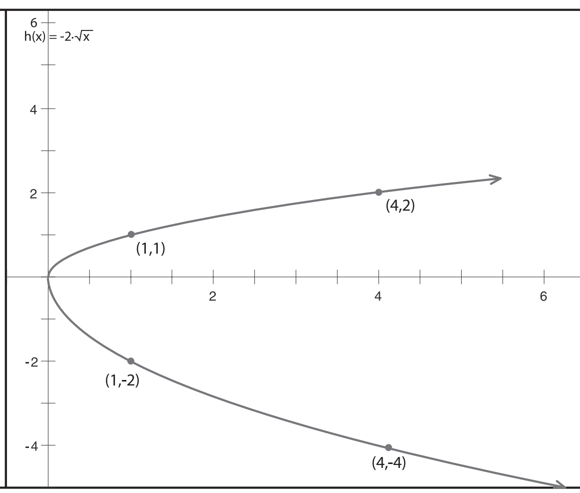
5.

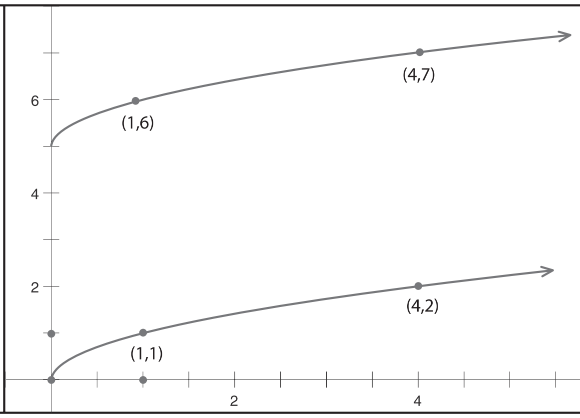
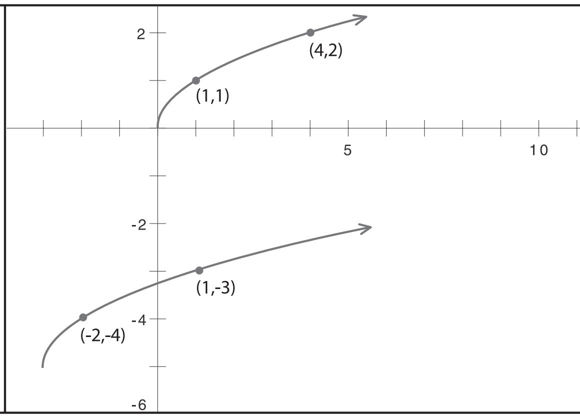
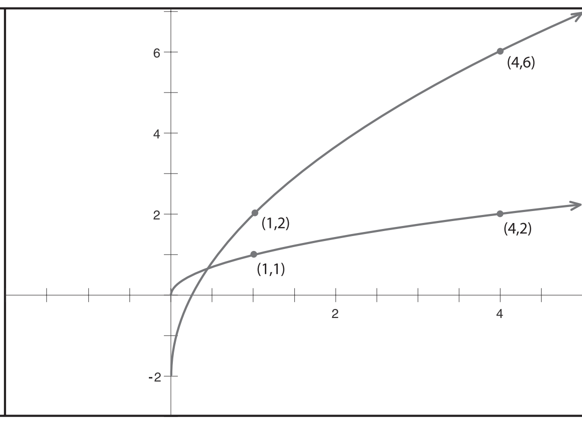


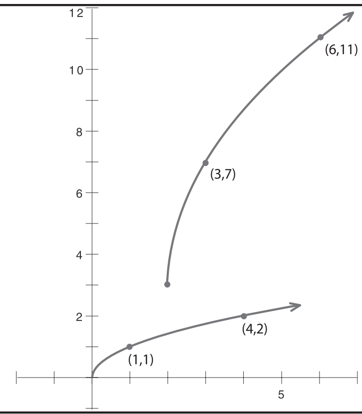
changed a and h | $a < 0$, $h > 0$

	Graph of function	Parameters changed	Changed parameter is positive or negative
6.		changed a, h, k	$a > 1$, $h < 0$, $k < 0$
7.		changed a, h, k	$a < -1$, $h < 0$, $k > 0$

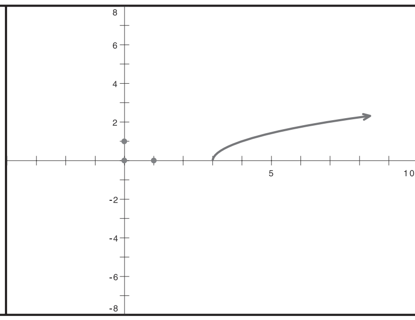
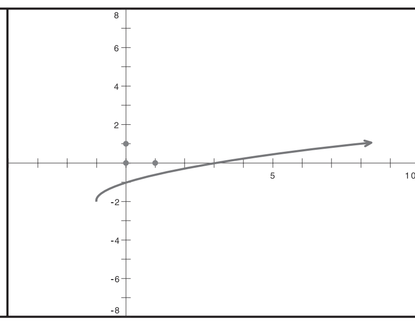
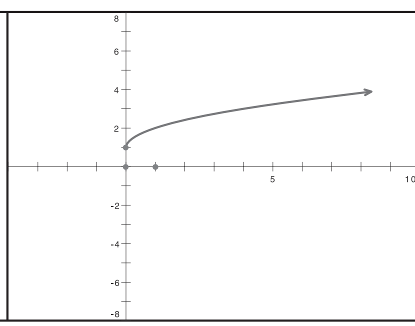
Activity Worksheet 2

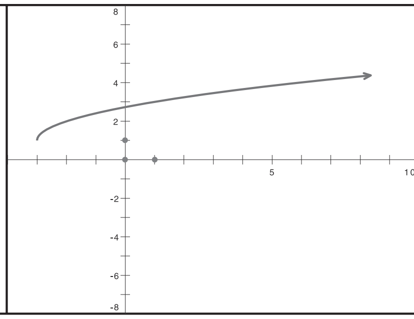
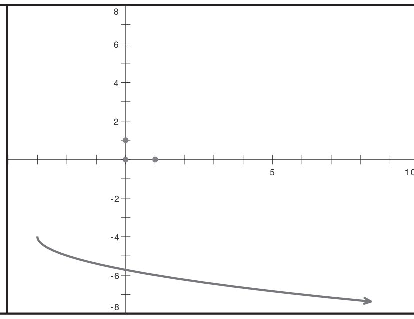
<table>
<tr><th></th><th>Function</th><th>Sketch the graph of the listed function on the given coordinate system. The parent graph is given.</th><th>Describe any transfor-mation that was used.</th><th>List the domain and range</th></tr>
<tr><td>1.</td><td>$y = \sqrt{x - 2}$</td><td>
</td><td>2 units to right</td><td>$x \geq 2$,
$y \geq 0$</td></tr>
<tr><td>2.</td><td>$y = \sqrt{x} - 2$</td><td>
</td><td>2 units down</td><td>$x \geq 0$,
$y \geq -2$</td></tr>
<tr><td>3.</td><td>$y = -2\sqrt{x}$</td><td>
</td><td>vertical stretch -2</td><td>$x \geq 0$,
$y \leq 0$</td></tr>
</table>

<table>
<tr><th></th><th>Function</th><th>Sketch the graph of the listed function on the given coordinate system. The parent graph is given.</th><th>Describe any transfor-mation that was used.</th><th>List the domain and range</th></tr>
<tr><td>4.</td><td>$y = \sqrt{x} + 5$</td><td>
</td><td>5 units up</td><td>$x \geq 0$,
$y \geq 5$</td></tr>
<tr><td>5.</td><td>$y = \sqrt{x + 3} - 5$</td><td>
</td><td>3 units left and 5 units down</td><td>$x \geq -3$,
$y \geq -5$</td></tr>
<tr><td>6.</td><td>$y = 4\sqrt{x} - 2$</td><td>
</td><td>vertical stretch of 4 and 2 units down</td><td>$x \geq 0$,
$y \geq -2$</td></tr>
</table>

7.	$y = 4\sqrt{x - 2} + 3$		2 units right, vertical stretch of 4 and 3 units up	$x \geq 2$, $y \geq 3$

Activity Worksheet 3

	Description of change to parent function	Graph	Rule of described function
1.	The parent graph is translated 3 units to the right.		$y = \sqrt{x - 3}$
2.	The parent graph is translated 1 unit to the left and 2 units down.		$y = \sqrt{x + 1} - 2$
3.	The parent graph is translated 1 unit up.		$y = \sqrt{x} + 1$

	Description of change to parent function	Graph	Rule of described function
4.	The parent graph is translated 3 units to the left and 1 unit up.		$y = \sqrt{x+3} + 1$
5.	The parent graph is translated 3 units to the left, reflected across the *x*-axis and translated 4 units down.		$y = -\sqrt{x+3} - 4$

Extension Questions:

- Create the rule for a square root function that opens in the negative *x* direction.

  One possible answer:

  $y=\sqrt{-x}$ *has domain values x ≤ 0. It is the reflection of the graph of* $y=\sqrt{x}$ *across the x-axis.*

- Can you have a function of the form $y=a\sqrt{(x-h)}+k$ that has a domain *x* ≤ *h* where *h* is any real number?

  The radicand must be greater than or equal to zero.

  $$x-h\geq 0$$
  $$x\geq h$$

  Thus, x may be equal to h but it cannot be less than h.

- Express $y=\sqrt{4x}$ in $y=a\sqrt{x-h}+k$ form and describe its relationship to the parent function.

  $$y=\sqrt{4x}$$
  $$y=2\sqrt{x}$$

  The function values of the parent function are multiplied by 2 to create this new function.

- Express $y=\sqrt{4x+16}$ in $y=a\sqrt{x-h}+k$ form. Describe its relationship to the parent function.

  $$y=\sqrt{4(x+4)}=2\sqrt{x+4}$$

  The parent function is translated to the left 4 units. The y-values of the new function are multiplied by 2 units.

Exponential Function Parameters

For the general exponential function, $f(x)=a \cdot b^x$, the initial value, *a*, and the growth/decay factor, *b*, are parameters whose values determine a particular exponential function. Four tasks follow that investigate the effects of these parameter changes.

Consider what happens to the graphs of the exponential function as the values of these parameters vary. In other words, how will changing the values of *a* or *b* affect domain and range, intercepts, graph shape or position, and asymptotic behavior?

1. Let $Y_1 = 2^x$ be the parent function for the general exponential function $f(x) = a \cdot b^x$.

   To examine the effects of changing *a*, you will let $a = \frac{1}{2}$, 2, and 4.

   Let

$$Y_1 = 2^x$$
$$Y_2 = \frac{1}{2} \cdot 2^x = \frac{1}{2} Y_1$$
$$Y_3 = 2 \cdot 2^x = 2Y_1$$
$$Y_4 = 4 \cdot 2^x = 4Y_1$$

   a. Indicate the calculator window you will use to graph all of the functions on the same grid.

   b. Sketch and label your graphs on the same grid.

   c. Describe what happens to the graphs of the exponential function as the values of *a* vary. In other words, how will changing the values of *a* affect domain and range, intercepts, graph shape or position, and asymptotic behavior of the parent function?

2. Let $Y_1 = 2^x$ be the parent function for the general exponential function $f(x) = a \cdot b^x$.

   To examine the effects of changing *a*, let $a = -\frac{1}{2}, -2$ and -4.

Let

$$Y_1 = 2^x$$
$$Y_2 = -\frac{1}{2} \cdot 2^x = -\frac{1}{2} Y_1$$
$$Y_3 = -2 \cdot 2^x = -2Y_1$$
$$Y_4 = -4 \cdot 2^x = -4Y_1$$

a. Indicate the calculator window for your graph.

b. Sketch and label your graphs on the same grid.

c. Describe what happens to the graphs of the exponential function as the values of *a* vary. In other words, how will changing the values of *a* affect domain and range, intercepts, graph shape or position, and asymptotic behavior of the parent function?

3. Now fix the value of *a* in $f(x) = a \cdot b^x$, and vary *b*.

   Let $a = 1$ and $b = \frac{1}{3}, \frac{1}{2}$, 2, and 3.

   Let

$$Y_1 = \left(\frac{1}{3}\right)^x$$
$$Y_2 = \left(\frac{1}{2}\right)^x$$
$$Y_3 = 2^x$$
$$Y_4 = 3^x$$

a. Indicate the calculator window for your graph.

b. Sketch and label your graphs on the same grid.

c. Describe what happens to the graphs of the exponential function as the values of *b* vary. In other words, how will changing the values of *b* affect domain and range, intercepts, graph shape or position, and asymptotic behavior of the parent function?

4. Either $a < 0$ or $a > 0$, and either $0 < b < 1$ or $b > 1$ in the general exponential function $f(x) = a \cdot b^x$.

   Let

$$Y_1 = -3\left(\frac{1}{2}\right)^x$$
$$Y_2 = -3(2^x)$$
$$Y_3 = 3\left(\frac{1}{2}\right)^x$$
$$Y_4 = 3(2^x)$$

a. Indicate the calculator window for a graph of these functions on the same grid.

b. Sketch and label your graphs.

c. Describe what happens to the graphs of the exponential function as the values of *a* and *b* vary. In other words, how will changing the values of *a* and *b* affect domain and range, intercepts, graph shape or position, and asymptotic behavior?

Notes

Materials:

Graphing calculator

Algebra II TEKS Focus: (2A.11) Exponential and logarithmic functions. The student formulates equations and inequalities based on exponential and logarithmic functions, uses a variety of methods to solve them, and analyzes the solutions in terms of the situation.

The student is expected to:

(B) use the parent functions to investigate, describe, and predict the effects of parameter changes on the graphs of exponential and logarithmic functions, describe limitations on the domains and ranges, and examine asymptotic behavior.

Additional Algebra II TEKS: (2A.1) Foundations for functions. The student uses properties and attributes of functions and applies functions to problem situations.

The student is expected to:

(A) identify the mathematical domains and ranges of functions and determine reasonable domain and range values for continuous and discrete situations.

Scaffolding Questions:

- In this activity you are focusing on the effects of parameter changes in $y = a \cdot b^x$. Think about other function families we have studied. What are the parameters in $y = mx + b$ and $y = a(x-h)^2 + K$? As you varied these parameters, how did the graphs change?
- Are you changing the parent function in any way that would affect the domain?
- What might cause the range of the parent function to change?
- What other representation could you explore to help you better see how the parent function is changing?
- What geometric transformations should you watch for?

Sample Solutions:

1. a. The calculator window for the graph shown is
   $-3 \le x \le 3,\ -2 \le y \le 8,\ Xscl = Yscl = 1$

   b. $Y_1 = 2^x$ is graphed in bold.

$$Y_2 = \frac{1}{2} \cdot 2^x = \frac{1}{2} Y_1$$
$$Y_3 = 2 \cdot 2^x = 2Y_1$$
$$Y_4 = 4 \cdot 2^x = 4Y_1$$

The graphs across the top from left to right are Y_4, Y_3, Y_1, and Y_2.

c. As we increase the value of $a, a > 0$, in $y = a \cdot b^x$,

- The domain and range do not change.
- The function is still an increasing function.
- The asymptotic behavior is still the same.
- The *y*-intercept is higher.
- The graphs all have the same shape because each of them may be rewritten as a power of 2.

$Y_2 = \frac{1}{2} \cdot 2^x = 2^{-1} \cdot 2^x = 2^{x-1}$

The graph of $Y_1 = 2^x$ has been shifted 1 unit to the right.

$Y_3 = 2 \cdot 2^x = 2^1 \cdot 2^x = 2^{x+1}$

The graph of $Y_1 = 2^x$ has been shifted 1 unit to the left.

$Y_4 = 4 \cdot 2^x = 2^2 \cdot 2^x = 2^{x+2}$

The graph of $Y_1 = 2^x$ has been shifted 2 units to the left.

Thus, each graph represents a horizontal shift of the original function. They have the same shape as the original function $Y_1 = 2^x$.

2. a. A possible calculator window for the graphs is

$$-3 \le x \le 3,\ -10 \le y \le 8,\ Xscl = Yscl = 1$$

b. The graph of $Y_1 = 2^x$ is shown in bold.

$$Y_2 = -\frac{1}{2} \cdot 2^x = -\frac{1}{2} Y_1$$
$$Y_3 = -2 \cdot 2^x = -2Y_1$$
$$Y_4 = -4 \cdot 2^x = -4Y_1$$

(2A.4) Algebra and geometry. The student connects algebraic and geometric representations of functions.

The student is expected to:

(A) identify and sketch graphs of parent functions, including linear ($f(x) = x$) quadratic ($f(x) = x^2$), exponential ($f(x) = a^x$), and logarithmic ($f(x) = \log_a x$) functions, absolute value of x ($f(x) = |x|$), square root of x ($f(x) = \sqrt{x}$), and reciprocal of x ($f(x) = 1/x$).

(B) extend parent functions with parameters such as *a* in $f(x) = a/x$ and describe the effects of the parameter changes on the graph of parent functions.

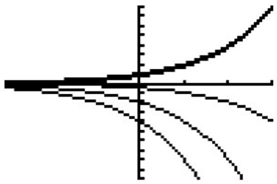
Connection to TAKS:

Objective 1: The student will describe functional relationships in a variety of ways.

Objective 2: The student will demonstrate an understanding of the properties and attributes of functions.

Objective 5: The student will demonstrate an understanding of quadratic and other nonlinear functions.

The graphs from top to bottom on the right are Y_1, Y_2, Y_3, and Y_4.



c. When we multiply b^x by a, $a < 0$, and a increasing in magnitude,

- The domain does not change.
- The asymptotic behavior does not change.
- The range is now $y < 0$ instead of $y > 0$.
- The function is now a decreasing function.
- The graph is now concave down.
- The y-intercept is negative and further from $y = 0$ as the magnitude of a, $| a |$, increases.
- The graphs all have the same shape because each of them may be rewritten as a power of 2.

$$Y_2 = -\frac{1}{2} \cdot 2^x = -2^{-1} \cdot 2^x = -2^{x-1}$$

The graph is the reflection of the graph of $Y_1 = 2^x$ has been translated 1 unit to the right and reflected across the x-axis.

$$Y_3 = -2 \cdot 2^x = -2^1 \cdot 2^x = -2^{x+1}$$

The graph is the reflection of the graph of $Y_1 = 2^x$ has been translated 1 unit to the left and reflected across the x-axis.

$$Y_4 = -4 \cdot 2^x = -2^2 \cdot 2^x = -2^{x+2}$$

The graph is the reflection of the graph of $Y_1 = 2^x$ has been translated 2 units to the left and reflected across the x-axis.

Thus, each graph represents a horizontal shift of the original function. They have the same shape as the original function $Y_1 = 2^x$.

3. a. A possible calculator window for the graphs is

$$-3 \le x \le 3,\ -2 \le y \le 8,\ Xscl = Yscl = 0$$

b. Let

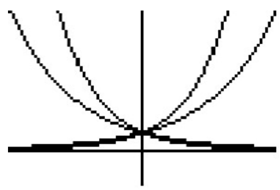
$$Y_1 = \left(\frac{1}{3}\right)^x$$

$$Y_2 = \left(\frac{1}{2}\right)^x$$

$$Y_3 = 2^x$$

$$Y_4 = 3^x$$

The graphs from left to right across the top are Y_2, Y_1, Y_4, and Y_3.



c. If $y = b^x$ and b is changed, then

- The domain and range do not change for the different values of b.
- The graphs all have y-intercept (0, 1) and no x-intercept.
- The graphs are concave up.
- The asymptotic behavior is still the same; the asymptote is $y = 0$.
- If $b > 1$, the function is increasing, while if $0 < b < 1$, the function is decreasing.

4. a. A possible calculator window for the graphs is

$$-3 \le x \le 3,\ -8 \le y \le 8,\ Xscl = Yscl = 0$$

b. Let

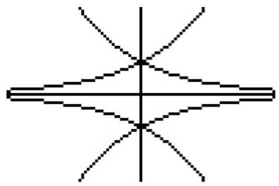
$$Y_1 = -3\left(\frac{1}{2}\right)^x$$

$$Y_2 = -3(2^x)$$

$$Y_3 = 3\left(\frac{1}{2}\right)^x$$

$$Y_4 = 3(2^x)$$

The graphs from left to right across the top are Y_3 and Y_4. The graphs from left to right across the bottom are Y_1 and Y_2.



c. Suppose that $Y_4 = 3 \cdot 2^x$ is the parent function for this set. Then

- All four functions have the same domain and asymptotic behavior.
- Y_3 is the reflection of Y_4 over the *y*-axis. It has the same *y*-intercept but is decreasing instead of increasing.
- Both graphs are concave up.
- Y_2 is the reflection of Y_4 over the *x*-axis.
- Y_2 has a *y*-intercept of (0, -3) instead of (0, 3). It is a decreasing function, and it is concave down instead of up.
- Y_1 is the reflection of Y_4 over the *y*-axis, followed by a reflection over the *x*-axis. This results in a *y*-intercept of (0, -3), and a function that is increasing and concave down.

Extension Questions:

- In problem 1, the *a* in $y = a \cdot b^x$ was actually a power of 2. What is the relationship between the graphs of the functions $y = 2^x$ and $y = 3 \bullet 2^x$?

  If you graph the two functions, they appear to have different shapes. However, we can express 3 as a power of 2 by using logarithms to solve the following equation.

$$2^k = 3$$
$$\ln 2^k = \ln 3$$
$$k \ln 2 = \ln 3$$
$$k = \frac{\ln 3}{\ln 2} \approx 1.58$$

Thus, $y = 3 \cdot 2^x$ *may be rewritten as a power of 2.*

$$y = 3 \cdot 2^x = 2^{1.58} \cdot 2^x = 2^{x+1.58}$$

This process may be used for any number that is multiplied by a power.

$$y = a \cdot b^x$$

a can be expressed as a power of b.

$$a = b^k$$
$$\ln a = k \ln b$$
$$k = \frac{\ln a}{\ln b}$$
$$a = b^{\frac{\ln a}{\ln b}}$$
$$y = a \cdot b^x = b^{\frac{\ln a}{\ln b}} \cdot b^x = b^{\frac{\ln a}{\ln b} + x}$$

- How did you investigate the effect of changing the value of *b*?

  We know that b has to be positive and cannot be 1, so in problem 3 we substituted integer values 2 and 3 and their reciprocals, $\frac{1}{2}$ *and* $\frac{1}{3}$. *If b is a fraction, we get a positive, decreasing function with the same y-intercept as the parent function. We could get the graph of* $y = \left(\frac{1}{2}\right)^x$ *by reflecting the graph of* $y = 2^x$ *over the y-axis.*

  If we use increasing integer values for b, we have the same y-intercept as the parent function, but the graph increases at a faster rate.

- Suppose *b* is any integer greater than 1. What is the connection between $Y_1 = b^x$ and $Y_2 = \left(\frac{1}{b}\right)^x$?

  The graph of Y_2 *is the reflection of the graph of* Y_1 *over the y-axis.*

- How can you show this algebraically?

$$Y_2 = \left(\frac{1}{b}\right)^x = \left(b^{-1}\right)^x = b^{(-x)}$$

If you have the graph of y = f(x), the graph of y = f(-x) is the reflection of the graph of y = f(x) over the y-axis.

Logarithmic Function Parameters

For the general logarithmic function $f(x) = \log_b\left(\frac{x}{a}\right)$, the value of *a* and the base *b* are parameters whose values determine a particular logarithmic function.

Describe what happens to the graphs of the exponential function as the values of these parameters vary. In other words, explain how changing the values of *a* or *b* affect domain and range, intercepts, graph shape or position, and asymptotic behavior.

1. Let *f(x)* = ln*(x)* be the parent function. Consider the following four functions.

$$Y_1 = \ln\left(\frac{x}{-2}\right)$$

$$Y_2 = \ln\left(\frac{x}{\frac{-1}{2}}\right)$$

$$Y_3 = \ln\left(\frac{x}{\frac{1}{2}}\right)$$

$$Y_4 = \ln\left(\frac{x}{2}\right)$$

a. Indicate the calculator window for your graph.

b. Sketch and label the graphs of the four functions on the same grid.

c. Describe what happens to the graphs of the logarithmic function as the values of *a* vary. In other words, how will changing the values of *a* affect domain and range, intercepts, graph shape or position, and asymptotic behavior of the parent function?

2. Consider the following four functions.

$$Y_1 = \log_{\frac{1}{10}}(x)$$
$$Y_2 = \log_{\frac{1}{e}}(x)$$
$$Y_3 = \ln(x)$$
$$Y_4 = \log_{10}(x)$$

a. Indicate the calculator window for your graph.

b. Sketch and label the graphs of the four functions on the same grid.

c. Describe what happens to the graphs of the logarithmic function as the values of *b* vary. In other words, how will changing the values of *b* affect domain and range, intercepts, graph shape or position, and asymptotic behavior of the parent function?

3. In the function $f(x) = \log_b\left(\frac{x}{a}\right)$, either $a < 0$ or $a > 0$, and either $0 < b < 1$ or $b > 1$.

Consider the functions below with different combinations of a and b, where $a = \pm 2$ and $b = \frac{1}{e}$ or e.

$$Y_1 = \log_e\left(\frac{x}{2}\right)$$

$$Y_2 = \log_{\frac{1}{e}}\left(\frac{x}{2}\right)$$

$$Y_3 = \log_e\left(\frac{x}{-2}\right)$$

$$Y_4 = \log_{\frac{1}{e}}\left(\frac{x}{-2}\right)$$

a. Indicate the calculator window for your graph.

b. Sketch and label the graphs of these four functions on the same grid.

c. Describe what happens to the graphs of the logarithmic function as the values of *a* and *b* vary. In other words, how will changing the values of *a* and *b* affect domain and range, intercepts, graph shape or position, and asymptotic behavior of the parent function?

Notes

Materials:

Graphing calculator

Algebra II TEKS Focus:
(2A.11) Exponential and logarithmic functions. The student formulates equations and inequalities based on exponential and logarithmic functions, uses a variety of methods to solve them, and analyzes the solutions in terms of the situation.

The student is expected to:

(B) use the parent functions to investigate, describe, and predict the effects of parameter changes on the graphs of exponential and logarithmic functions, describe limitations on the domains and ranges, and examine asymptotic behavior.

Additional Algebra II TEKS:
(2A.1) Foundations for functions. The student uses properties and attributes of functions and applies functions to problem situations.

The student is expected to:

(A) identify the mathematical domains and ranges of functions and determine reasonable domain and range values for continuous and discrete situations.

Scaffolding Questions:

- What is the graph of the parent function $f(x) = \ln(x)$?
- In this activity you are focusing on the effects of parameter changes in $y = \log_b\left(\frac{x}{a}\right)$. What are these parameters?
- If you are using your graphing calculator to graph logarithmic functions, what bases are available for you to use?
- What theorem for logarithms will allow you to rewrite $Y_1 = \log_{\frac{1}{10}}(x)$ so that you can graph it on your calculator?
- What other properties of logarithms might help you in this investigation?
- Are you changing the parent function in any way that would affect the domain?
- What might cause the range of the parent function to change?

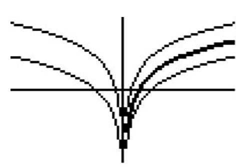
Sample Solutions:

1. a. A possible calculator window for the graphs is

$$-8 \le x \le 8,\ -3 \le y \le 3, xscl = yscl = 0.$$

b. Clockwise from left to right the graphs are

Y_1, Y_2, Y_3, $Y = \ln(x)$ (in bold) and Y_4



c. We know that $y = ln(x)$ is the parent function.
As we increase the value of a, $a > 0$, in $y = \log_b\left(\frac{x}{a}\right)$

- The domain and range do not change.
- The function is still an increasing function.
- The graph is still concave down.
- The asymptotic behavior is the same.
- As a increases, the x-intercept moves more to the right of the origin.
- The graph is less steep.

In Y_1 and Y_2, the variable x is divided by a negative number. This causes a reflection of the parent graph over the y-axis because x must be negative so that $\frac{x}{a}$ is positive.

- The range is still the same.
- The graph is still concave down.
- The asymptotic behavior is still the same.
- The domain is now the negative real numbers.
- The function is now a decreasing function.
- The steepness of the graph is determined by the magnitude of a, $|a|$. As the magnitude increases, the graph is less steep.

2. To graph the functions the change of base theorem can be used.

$$\log_a b = \frac{\ln b}{\ln a}$$

Therefore,

$$Y_1 = \log_{\frac{1}{10}}(x) = \frac{\ln x}{\ln\left(\frac{1}{10}\right)}$$
$$Y_2 = \log_{\frac{1}{e}}(x) = \frac{\ln x}{\ln\left(\frac{1}{e}\right)}$$
$$Y_3 = \ln(x)$$
$$Y_4 = \log_{10} x$$

(2A.4) Algebra and geometry. The student connects algebraic and geometric representations of functions.

The student is expected to:

(A) identify and sketch graphs of parent functions, including linear ($f(x) = x$), quadratic ($f(x) = x^2$), exponential ($f(x) = a^x$), and logarithmic ($f(x) = \log_a x$) functions, absolute value of x($f(x) = |x|$), square root of x($f(x) = \sqrt{x}$), reciprocal of x($f(x) = 1/x$).

(B) extend parent functions with parameters such as a in $f(x) = a/x$ and describe the effects of the parameter changes on the graph of parent functions.

(C) describe and analyze the relationship between a function and its inverse.

Connection to TAKS:

Objective 1: The student will describe functional relationships in a variety of ways.

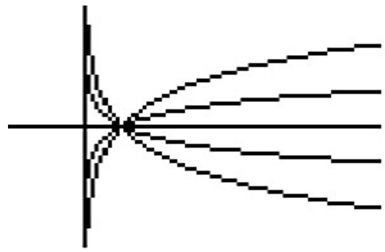
Objective 2: The student will demonstrate an understanding of the properties and attributes of functions.

Objective 5: The student will demonstrate an understanding of quadratic and other nonlinear functions.

a. A possible calculator window for the graphs is

$$-2 \le x \le 8, \ -3 \le y \le 3, \ xscl = yscl = 0$$

b. The graphs from the right, top to bottom, are Y_3, Y_4, Y_1 and Y_2.



c.

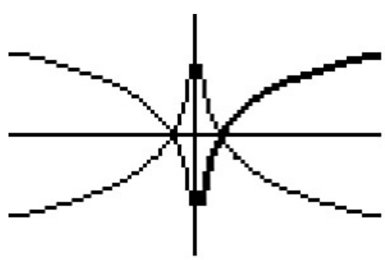
- The domain and range are the same.
- The *x*-intercept is the same.
- Asympotic behavior is the same.
- As *b* increases in magnitude, the graph becomes less steep.
- If 0 < *b* < 1, the function decreases instead of increases and is concave up instead of concave down.
- If 0 < *b* < 1, the graph is a reflection of the parent graph over the x-axis. For example, if $b = \frac{1}{2}$, then

$$\log_{\frac{1}{2}}(x) = \frac{\ln(x)}{\ln\left(\frac{1}{2}\right)} = \frac{\ln(x)}{\ln(1) - \ln(2)} = \frac{\ln(x)}{0 - \ln(2)} = -\frac{\ln(x)}{\ln(2)}$$

3. a. A possible calculator window for the graphs is

$$-8 \le x \le 8, \ -3 \le y \le 3, \ xscl = yscl = 0$$

b. The bold graph is Y_1. Moving clockwise from there around the outer edge, the other 3 graphs are Y_2, Y_4, and Y_3, respectively.



c. Comparing Y_1 and Y_2 :

- Only the parameter *b* changes. For the parent function *b* = *e*, and for the transformed function $b = \frac{1}{e}$.
- The graph of Y_2 is the reflection of Y_1 over the *x*-axis.
- The functions have the same domain, range, *x*-intercept, and vertical asymptote.
- The parent function is increasing without bound. Y_2 is decreasing without bound.
- Y_1 is concave down. Y_2 is concave up.

Comparing Y_1 and $Y_{3:}$:

- Only the parameter *a* changes. For Y_1, *a* = 2 and for Y_3, *a* = -2.
- The graph of Y_3 is the reflection of Y_1 over the *y*-axis.
- The functions have the same range and vertical asymptote, and both are concave down.
- The domain of Y_3 is $x < 0$ instead of $x > 0$, and the *x*-intercept is (-2, 0) instead of (2, 0).
- Y_1 is increasing without bound. Y_3 is decreasing without bound.

Comparing Y_1 and Y_4 :

- Both parameters *a* and *b* change. For Y_1, *b* = *e* and *a* = *1*. For Y_4, $b = \frac{1}{e}$ and *a* = -2.
- The graph of Y_4 is the reflection of Y_1 over the *x*-axis and then over the *y*-axis.
- The functions have the same range and vertical asymptote, and both functions increase without bound.
- For Y_4, the *x*-intercept is (-2, 0) instead of (2, 0).
- Y_1 is concave down. Y_4 is concave up.

Extension Questions:

- For which function, $y = a \cdot b^x$ or $y = \log_b\left(\frac{x}{a}\right)$, is it easier to analyze the effects of changes in the parameters *a* and *b*?

  The exponential function is easier to analyze.

- How are the two functions related? How might this have helped you analyze the logarithmic functions?

  They are inverse functions. The graph of the logarithmic function is the reflection of the graph of the exponential function over the line y = x . This switches domain and range.

  The y-intercept becomes an x-intercept. A horizontal asymptote becomes a vertical asymptote. If the exponential function is increasing, so is its inverse. If the exponential function is decreasing, so is its inverse. Concavity switches.

- What about graph steepness and location of *y*-intercepts?

  Increasing the magnitude of a and b in an exponential function made its graph steeper. Similar changes in a and b in logarithmic functions would make its graph less steep.

  Increasing the magnitude of a in an exponential function moved its y-intercept further from the original. Similar changes in logarithmic functions moves its x-intercept further from the origin.

- Consider the functions in Task A. How could properties of logarithms help you compare $y = \ln(x)$, $Y_3 = \ln(2x)$, and $Y_4 = \ln\left(\frac{x}{2}\right)$?

  $$Y_3 = \ln(2x) = \ln(x) + \ln(2)$$

  $$Y_4 = \ln\left(\frac{x}{2}\right) = \ln(x) - \ln(2)$$

  Since $\ln(2) > 0, \ln(x) - \ln(2) < \ln(x) < \ln(x) + \ln(2)$. *This shows us that* $Y_4 < \ln(x) < Y_3$.

- Can you apply properties of logarithms to $Y_1 = \ln\left(\frac{x}{-2}\right)$ and $Y_2 = \ln(-2x)$?

  Yes. For example, ln(-2x) = ln(2 · -x) = ln 2 + ln(-x). Since x < 0 , -x > 0 and therefore, ln(-x) is defined.

- In this investigation, the function representations that you used were graphing and perhaps tables. Could you have described the transformations on these functions analytically? If so, how?

Yes. We can do this analytically, if we understand how changing a function's parameters transforms the parent function and we apply properties of logarithms to the functions we are investigating.